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By

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FTD-TT-62-132/1

UNEDITED ROUGH DRAFT TRANSLATION

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English Pages: 109

SOURCE: Zhurnal Prikladnoy Mekhaniki i Tekhnicheskoy Fiziki, No. 4, 1961, pp. 3-56.

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THE MATHEMATICAL THEORY OF EQUILIBRIUM CRACKS FORMED IN BRITTLE FRACTURE

By G.N. Barenblatt (Moscow)

This article gives an account, from a single general point of view, of the basic problem formulations in the theory of equilibrium cracks and of the results obtained in this theory.

The theory of cracks is a rather new field of mechanics and, as a result, there are no monographs containing surveys of this problem. Consequently, it seemed advisable to present the principles of this theory in greater detail.

The first and second sections of this article consistute an introduction to and a brief outline of the development of the theory of equilibrium cracks. The third section considers the structure of the edge of the equilibrium crack in a brittle solid. The fourth section presents the basic hypotheses and gives a general formulation of the problem of equilibrium cracks; experimental confirmations of this theory of cracking are considered. The fifth section deals with a number of specific problems of the theory of equilibrium cracks; problems of resistance to cracking are considered. Finally, the sixth section deals with the problem of wedging, which is important for the theory of cracks, and briefly considers the results obtained which have a bearing on the dynamics of cracking.

In writing this article, the author has attempted to avoid repetition of available surveys of various aspects of brittle fracture. By its nature, this survey properly touches on the theory of cracking, which is the mathematical theory of brittle fracture. In this connection, the voluminous available experimental reports are cited only insofar as they are necessary for confirmation of the theory advanced and for determining its limits of applicability. In contrast to the mathematical theory, experimental studies of brittle fracture are not once considered in the appropriate surveys and monographs. In addition, these sources ignore or hardly treat at all of problems related only to mathematical techniques for solving problems of elasticity theory. Nor do they deal with the formation of cracks. In attempting to assemble all accounts from a unified point of view. the author has occasionally permitted deviation from the original treatises in citing isolated specific results obtained by other investigators.

The author is indebted to Ya.V. Zel'dovich and Yu.N. Rabotnov (Academy of Sciences USSR) and S.S. Grigoryan of the MGU (Moscow State University) for their unflagging interest and attention to his work on cracks and for a number of valuable suggestions. He remembers with gratitude his helpful discussions with S.A. Khristianovich (Academy of Sciences USSR). The author considers it his duty to express his thanks to Professor G. Kyurt', the editor-in-chief of the publication Advances in Applied Mechanics, and Professor G.G. Chernyy (MGU) for their obliging assistance in writing this article. The author also notes with gratitude the help of I.A. Markuzon in compiling the bibliography for the present survey.

1. Introduction

The subject matter of the theory of equilibrium cracks is the study of equilibrium in solids containing cracks.

Let us consider a solid which contains cracks (Fig. 1) and

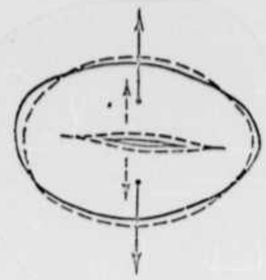

Fig. 1.

which is in equilibrium under the action of some system of loads.

The solid is considered to be capable of withstanding any finite stresses and to be ideally brittle, i.e., to retain the property of linear elasticity to the fracture point; the feasibility of using an ideally brittle solid as a model for a real material will be considered below.

The spread of the crack (the distance between the opposing crack surfaces) is always much less than the length of the crack. Consequently, cracks can be regarded as surfaces at which disruption of the continuity of the material occurs, i.e., surfaces at which discontinuities of the shear vector occur.

Unless otherwise noted, we shall deal below with the two-dimensional normal-tensile-fracture cracks, i.e., portions of a plane which are bounded by closed curves (the boundaries of the cracks) and are subject to fracture only along the normal components of the shear vector. We may deal with the case where fracture occurs along the tangential slip components on the fracture surfaces of an ideally brittle body in the same fashion as for normal-rupture cracks.

It might be supposed that research on the equilibrium of elastic bodies containing cracks could be carried out by the general methods of elasticity theory, as is done for bodies containing cavities (Fig. 2). However, there is a basic difference between these two problems. Even when the loads acting on the body are varied considerably, the shape of the cavities changes only slightly. At the same time, cracks whose surfaces also form a section of the boundary of the solid can be widened sharply with even a small increase in the load acting on the body (see Figs. 1 and 2, where the broken lines denote the additional loads and the corresponding positions of the boundaries of the body).

Thus, one of the basic premises of the classical linear theory of elasticity is not satisfied for problems in the theory of cracking, namely the assumption that the change in the boundaries of the body under load is small, which makes it possible to assume that the boundary conditions are observed on the surfaces of a nondeformed bedy. This makes the problem of equilibrium in a body containing cracks essentially nonlinear, in contrast to the traditional problems of elasticity theory. In problems of crack theory, it is necessary to determine from the equilibrium conditions not only the distribution of stresses and strains but also the limits of the region for which the equilibrium equations can be solved.

As we know, nonlinear problems of this type ("problems with unknown limits") have already long been encountered in various branches of mathematical physics. It is sufficient to note the theory of the jet and the theory of finite-amplitude waves in hydrodynamics, the theory of flow past a body in the presence of shock waves in gas dynamics, Stefan's law of freezing in the theory of heat transfer, etc. The principal difficulty in these problems is associated with finding the limits of the region in which the solution is sought. The location of the crack surfaces for a given applied load presents exactly the same basic problem in the theory of equilibrium cracks.

The differential equilibrium equations and the usual boundary conditions of elasticity theory are fundamentally unable to provide solutions to these problems without consideration of additional factors. This may be seen from the fact that it is formally possible to set up a solution for the equations which would satisfy the usual boundary conditions without even specifying the crack surfaces. Analysis of the formal solutions obtained in this case shows that, generally speaking, the tensile stresses σ normal to the surface of a crack are infinite on the circumference of the crack according to these solutions. More precisely, near an arbitrary spot on the circumference of the crack

$$\sigma = \frac{N}{\sqrt{s}} + \text{a finite quantity.} \qquad (1.1)$$

Here s is the distance of a point of the body lying in the plane of the crack from the circumference of the crack; N is the "coefficient of stress intensity," whose magnitude depends on the applied loads, the shape of the crack outline, and the coordinates of the point of this outline being considered, but is independent of s. Here the form of the normal section of the deformed surface of the crack near its edge is unnaturally rounded (as in Fig. 3 or somewhat differently; see detailed discussion below).

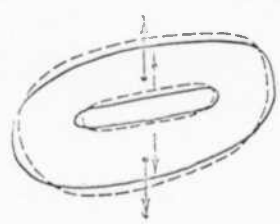

Fig. 2.

Fig. 3. 1) r.

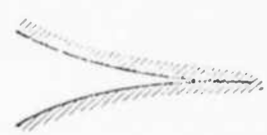

Fig. 4.

Generally speaking, however, there exist exceptional crack contours for which the stresses at the edges of the cracks are finite (N = 0) with a given load and the opposing surfaces of the cracks unite smoothly at the boundaries, so that the shape of the section of the crack surfaces near the edge is of the form shown in Fig. 4. It is possible to show that for such contours (and only for such contours) the energy liberated with a small change in the contour of the crack in the vicinity of a given point equals zero. Hence it follows that equilibrium cracks can be bounded only by such contours.

Thus, if all the loads acting on a body are given, the problem of the theory of equilibrium cracks is formulated in the following fashion. For a given distribution of the original cracks and a given system of forces acting on the body, it is necessary to determine the stresses, deformations, and crack contours for the elastic body under consideration so as to satisfy the differential equilibrium equations and boundary conditions, and to ensure that the stresses are finite or, which is the same thing, that the opposing banks of the crack outline unite smoothly. If the location of the initial cracks is not specified, the problem presented has a multivalued solution, since by virtue of the model adopted, the body can withstand any finite stresses. This is natural, since one and the same load on the same body may produce no cracks at all, one crack, two cracks, etc.

In the general case of curved cracks, their form is determined not only by the load existing at the moment in question, but also by the history of the process by which the body was loaded. However, although the symmetry of the body and the monotonically increasing applied loads ensure the development of surface cracks, the contours of these cracks will be determined solely by the load acting at the time. All results available in crack theory at the present time correspond to particular cases of this simplified statement of the problem.

Generally speaking, it is necessary to include more than simply the load applied to a body in the system of forces acting on the body. This is shown by the following example. Let us attempt to determine the contour of an equilibrium crack in the case of the load depicted in Fig. 1. If, in accordance with the usual methods of elasticity theory, we assume the surface of the crack to be free of stress, as is the case for the surface shown in Fig. 2, we obtain a paradoxical result; however we select the contour of the crack, the tensile stress at its edge is always infinitely large. Consequently, there are no equilibrium cracks: at as small a fracture stress as you please, a body having a crack breaks in two!

This obvious conflict with reality can be explained very simply. Having primitively used a model of an elastic body, we did not study all the forces acting on the body. In order to construct an adequate theory of cracking, it has proven necessary — and this is one of the main differences between the problems of the theory of cracking and the traditional problems of elasticity theory — to consider the molecular cohesive forces acting in the vicinity of the crack contour where the distance between the opposing faces of the crack is small and they attract one another powerfully.

Although, in principle, consideration of cohesive forces solves the problem, it seriously complicates research. The difficulty lies in the fact that neither the distribution of cohesive forces over the surface of a crack nor even the dependence of the intensity of these forces on the distance between the opposing faces of the crack is known. In addition, the distribution of cohesive forces depends in general on the loads applied. However, if the cracks are not too small, there is a way out of this difficulty. The fact is that the intensity of the cohesive forces very rapidly reaches a high maximum approximating Young's modulus when the distance between the opposing faces of the cracks is increased and then rapidly decreases after passing this maximum. Consequently, we can adopt two simplifying hypotheses.

The first of these is that the area of the section of the crack surface on which the cohesive forces act can be assumed negligibly small in comparison with the total area of the crack surface.

According to our second hypothesis, the shape of the crack surface (and consequently, the local distribution of cohesive forces) in the vicinity of the points on the contour of the crack at which the cohesive forces are at maximum intensity does not depend on the applied load.*

For example, the cohesive forces are at their maximum possible intensity for a given material under a given set of conditions at all points on the contour of a crack formed in the primary fracture of the material while the load is increased. For the majority of real materials under ordinary conditions, cracking is irreversible. If an irreversible crack is formed with the help of an artificial notch and without subsequent expansion or if it is produced on a reduction in the load from a crack that existed under a heavy load, the intensity of the cohesive forces at the contour of the crack will be less than the maximum possible value. The cohesive forces acting on the surface of a crack compensate applied fracture loads and ensure finite stresses and a smooth junction between the faces of the crack. With increasing fracture loads, the cohesive forces increase, adapting themselves in this sense to the increasing tensile stresses. In this case, the crack does not widen further until the maximum possible intensity of cohesive forces at the contour of the crack is achieved. Only when the maximum possible intensity of the cohesive forces is achieved at its contour does the crack begin to develop.*

The gradual development of the edge of a crack as the tensile load is increased is shown schematically in Fig. 5.

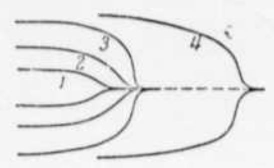

Fig. 5.

If we use the first of the hypotheses given above, the molecular cohesive forces enter the picture under the conditions determining the position of the contours of the cracks only in the form of the integral

$$K = \int_0^d \frac{G(t)\,dt}{\sqrt{t}} \quad (1.2)$$

Here G(t) is the intensity of the cohesive forces acting in the vicinity of the crack margins, t is the distance along the surface of the crack, reckoned along the normal to its contour, and d is the width of the region in which the cohesive forces act. For points on the contour to which the second hypothesis is applicable, this integral is, for a given set of conditions (temperature, composition and pressure of surrounding atmosphere, etc.), a constant for a given material and determines its resistance to the formation of cracks. It can be shows that the value of K is simply related to the surface tension T_0 of the material, the modulus of elasticity E, and Poisson's ratio ν:

$$K^2 = \frac{\pi E T_0}{1 - \nu^2} \quad (1.3)$$

Further, for all points on the contour of a crack at which the intensity of cohesive forces is at a maximum, the coefficient of stress intensity N, which occurs in (1.1) and is calculated without considering cohesive forces, equals K/π.

For all points on the contour of a crack at which the intensity of cohesive forces has not reached the maximum, the coefficient of stress intensity without considering the cohesive forces is less than K/π.

The considerations cited above clarify the manner in which cohesive forces manifest themselves in this problem enough for us to formulate the basic problem of the theory of equilibrium cracks.* When the symmetry of the body, the initial cracks and the monotonically increasing applied loads ensure development of a system of plane cracks, this problem is stated in the following form.

Let the original cracks in the body have a certain system of contours. It is necessary to find the stress and shear field corresponding to the load in question, and the system of contours of the surface cracks which surround the contours of the initial cracks (and perhaps are coincident with the original cracks to some extent).

Mathematically, the problem reduces to the synthesis of a system of contours in which the intensity coefficient N of the fracture stress calculated without considering the cohesive forces at all points on the contours not lying on the contours of the original cracks equals K/π and does not exceed K/π at all points on the contours lying on the contours of the initial cracks.

The proposed formulation of the problem eliminates direct consideration of the molecular cohesive forces (they enter the problem only through the constant K). Consequently, the stress and deformation field defined by the solution to this problem will not correspond to actuality when we are dealing with a rather small region around the contours of the cracks.

It is obvious that when the cracks are reversible or when the applied load is sufficiently large that the contours of all cracks lie outside the contours of the initial cracks, the form of the latter ceases to have significance.

The equilibrium state which corresponds to the maximum possible intensity of the cohesive forces, even at only one point on the contour of the crack, may be stable or unstable. Depending on this, further growth of the crack under increased loads may proceed by various methods. When the equilibrium is stable, a slow increase in stress causes a slow, quasistatic transition of the crack from one equilibrium state to another. If equilibrium is unstable, the crack begins rapid dynamic growth at the slightest increase in the load over the equilibrium value. In some cases, when there are no neighboring stable equilibrium states, this leads to complete fracture of the body. The development of the theory of cracks was such that, until recently, the chief considerations were problems of precisely the latter type and, consequently, the beginning of crack growth was occasionally identified with complete fracture of the body. It is necessary to realize clearly that the situation in which this actually obtains is a particular case and its practical value must not be exaggerated.

After a brief sketch of the development of the mathematical theory of cracks, we shall set forth below the general foundations of the theory of equilibrium cracks and the results of solving the most characteristic specific problems of this theory that have been dealt with up to the present time. At the end of the article, we shall consider briefly the dynamic problems of the theory of cracking.

II. Outline of Development of Theory of Equilibrium Cracks

Research in the area of the theory of cracks was begun nearly fifty years ago with the work of Inglis [1]. Within the framework of the classical theory of elasticity, this work solved the problem of the equilibrium of an infinite body with an isolated elliptical cavity (in particular, with a rectilinear slit) in a homogeneous stress field. The work of N.I. Muskhelishvili [2], which was also within the framework of classical elasticity theory, provided a simpler and more effective solution to the problem of equilibrium of an infinite body with an elliptical cavity in an arbitrary stress field.

However, despite the great value of References [1] and [2] for subsequent research, they still did not set up a true theory of cracks. The solutions obtained by these works had two properties which it is difficult to explain.

First of all, the length of a crack at a given load was found to be indeterminate; a solution could be constructed using any value of this parameter. At the same time, everyday experience suggested that the size of cracks appearing in a body was somehow related to the tensile loads applied to the body. When the load is increased, cracks already in the body do not begin to widen at first, as the load was small; when a certain stress was reached, they begin to widen, and to do so in different ways depending on the method by which the load is applied. In some cases, the cracks grow rapidly until sufficient to fracture the body while the load was maintained constant, while in other cases the cracks grew slowly and ceased to widen as soon as the load stopped increasing. Further, since the spread of the crack is generally small in comparison with its length, it is natural to represent the crack in the form of a slit. Thus, in this case, the tensile stresses at the ends of the crack prove to be infinitely great in Inglis' problem; generally speaking, this was also true of the problem considered by N.I.

Muskhelishvili. It is clear that solutions in which infinitely great tensile stresses are obtained at the edge of the crack are unsuitable for any physically correct model of a brittle body.

Thus, the direct application of the classical system of elasticity theory to the problem of cracks led to a statement of the problem which was incomplete and gave physically inapplicable solutions.

The work of Griffiths [3, 4] is correctly regarded as basic for the theory of cracks in brittle fracture. These introduced for the first time the important idea that to develop an adequate theory of cracks, it would be necessary to perfect a suitable model of the brittle body by introducing the molecular cohesive forces acting in the neighborhood of the edge of the crack.

Griffiths investigated the following problem. In an infinite brittle body under tension at infinity by a uniform stress P_0, let there be a rectilinear crack of definite size $2\underline{l}$. It is necessary to determine the critical value P_0 of the stress at which the crack will begin to widen.

Griffiths dealt with the molecular cohesive forces as forces of surface tension which were forces interior to the given body; he disregarded their action on the stress deformation field.

With this condition, the change ΔF in free energy ("the total potential energy" according to Griffiths' terminology) of a brittle body containing a crack in comparison with the same body subject to the same loads but without a crack equals the difference between the surface energy U of the crack and the decrease W in the elastic energy of the body caused by the formation of the crack. In order for this crack to grow, it is necessary that an increase in the size $2\underline{l}$ of the crack not cause an increase in the body's free-energy change ΔF. Thus, the parameters of the critical, equilibrium state are found from the condition

$$\frac{\partial (U - W)}{\partial l} = 0 \qquad (2.1)$$

However, the surface energy U of the crack equals the product of the surface area of the crack by the energy T_0 consumed in the formation of a unit area of the crack. The magnitude of T_0, the surface tension, may then be assumed constant for a given material under a given set of conditions when certain rather general assumptions are made. Consequently, Griffiths' determination of the critical load reduces to finding the magnitude of $\partial W/\partial l$ ("the rate of liberation of elastic energy"). For the simplest case as studied by him, Griffiths calculated this value using the results of Inglis [1] and obtained expressions for the critical values of the fracture stress in the form

$$p_0 = \sqrt{\frac{2ET_0}{\pi(1-\nu^2)l}}, \quad p_0 = \sqrt{\frac{2ET_0}{\pi l}} \qquad (2.2)$$

for conditions of plane deformation and a plane stressed state, respectively.

In the theoretical part of this work, Griffiths also obtained results with a bearing on research on the structure of a crack near its ends. Griffiths conducted this research on the basis of the classical solution of elasticity theory, which was arrived at without considering cohesive forces. In this case, it is natural that if the crack is regarded as a slit, the tensile stresses at the ends of the crack will be infinitely large. In order to eliminate this infinite tensile stress at the ends of the crack, Griffiths made an attempt to improve his description of the crack, considering it to have an elliptical cavity with a finite radius of curvature ρ at its end (Fig. 3). However, according to his estimates, the radius of curvature at the end of the crack was of the order of the interatomic distances, and this obviously proved the incorrectness of the approach: in any investigation based on the concept of a continuous medium, distances of the order of interatomic distances cannot be considered finite.

This section of Griffiths' work is flawed by the following: despite the fact that for definite equilibrium sizes of the crack it is possible to neglect the effect of molecular cohesive forces on the field of stresses and deformations, it is impossible to do so in research on the structure of a crack in the vicinity of its ends. The order of distances at which cohesive forces have an effect compares with the distances over which the shape of the crack essentially varies. To a considerable degree, therefore, Griffiths' analysis of the structure of the crack edges cannot be acknowledged as correct. In particular, as will be shown in detail, Griffiths' conclusion regarding the rounded form of a crack near its end is incorrect.

This aspect of the matter, which is obviously of basic importance, has remained unclear until recently and in many cases, has led to incorrect interprepation of Griffiths' results [5].

In addition to the basic deficiency noted here, there are several inaccuracies in the calculations in the theoretical sections of Reference [3]. Soon after the publication of this work, Smekal [6] published a detailed commentary on it which also contained a very interesting general discussion of the problem of brittle fracture and corrected the aforementioned inaccuracies. The later work of Wolf [7] gave a clearer and simpler account of Griffiths' results and also carried out analogous calculations for somewhat different (but also homogeneous) stressed states. Reference

[7] also dealt with the connection between Griffiths' theory of fracture and theories of strength which had been proposed previously.

The report by L.V. Obreimov [8] in connection with his experiments on the cleavage of mica, investigates the tearing away of a thin chip from a body by a splitting wedge slipping along its surface and touching the chip at one point. Using the approximate methods of the theory of thin beams, and referring to the analogous work of Griffiths on the energy approach, I.V. Obreimov formulated an expression which related the shape parameters of the crack to the surface tension. Reference [8] was later supplemented by many researchers [9-12].

The determination of the rate of liberation of elastic energy $\partial W/\partial \underline{l}$ for tensile-stress fields more complex than the homogeneous field or for other crack configurations encountered considerable mathematical difficulties. Research by Westergaard [13], Sneddon [14, 15], Snedden and Elliot [16], and Williams [17] ascertained the distribution of stresses and strains in the vicinity of shearing-fracture surfaces. In addition to the classical works by Muskhelishvili [2, 18, 19], the research of Westergaard and Snedden constituted a mathematical basis for subsequent work on the theory of cracks. However, no equilibrium conditions for new particular cases, not to mention any general case of loading, were derived in these studies.

The works of Sack [20], Willmore [21], and Bowie [22] gave the equilibrium conditions for certain new particular cases of loading and crack location. The direct application of the energy method in these works required surmounting considerable difficulties in calculation. Because the equilibrium states in the problems considered in References [20-22] were unstable and unique, the equilibrium conditions for them coincide with the conditions for total fracture of the body.

An important step for the theory of cracks was the work done by Irwin [23] and Orowan [24] in which the concept of quasibrittle fracture was developed. Irwin and Orowan drew attention to the fact that a number of materials which showed high plasiticity in standard tensile tests fractured according to a "quasibrittle" mechanism when cracks were formed. This meant that the developing plastic deformation was concentrated in a very narrow layer in the vicinity of the surface of the cracks. As Irwin and Orowan showed, it is possible to use Griffiths' theory of brittle fracture in these cases, replacing the surface tension by the effective surface-energy density. In addition to the specific work of fracturing internal bonds (surface tension) this quantity includes the specific work expended on plastic deformation in the layer of the crack near the surface, which occasionally exceeds the surface tension by several orders.

The introduction of quasibrittle fracture considerably broadened the area of application of the theory of brittle fracture, and without doubt served for a while as one of the basic reasons for the recent revival of interest in this problem. Irwin, Orowan, and other authors published a number of papers [23-32] devoted to the development of a generalized theory of brittle fracture, research on the limits of its applicability, and analysis of experimental data from the viewpoint of this theory. It is necessary to mention the article by Bueckner [33], in which a very general energy analysis of brittle and quasibrittle fracture was given on the basis of the theoretical system of Griffiths, Irwin, and Orowan.

In all the enumerated works, there remained unexplained the problem of the structure of a crack in the vicinity of its contour. In a very interesting study [34] devoted to physicochemical analysis of the deformation process, P.A. Rebinder for the first time expressed the idea of the wedge-shaped form of the crack ends and the necessity for a corresponding improvement of Griffiths' theory. In analyzing crack shapes, Elliot [35], Mott [36], and Ya.I. Frankel' [5] proceeded from the notion of a crack of infinite length between two unbroken blocks of the material being fractured, separated by the normal interatomic distance before the crack is formed.

In Reference [35], the blocks were regarded as semi-infinite. Proceeding from the classical solution to the problem of elasticity theory for rectilinear [1] and discoid [20] cracks of size 2c in a uniform fracture stress field p, [35] gives calculations for the distribution of normal stresses σ_y and the transverse shears v of points on planes lying at a distance of one half the normal interatomic distance from the crack surfaces. The function σ_y (2v), which contains p and c as parameters, was identified with the expression for the molecular cohesive forces as a function of distance; integration of this function gave the surface stress, which is thus related to p and f. The author identified the relation obtained with the fracture condition; this condition naturally differed from that of Griffiths. The distribution found for the transverse shears was identified with the form of the crack.

This approach was unsatisfactory for the following reasons. The formal use of the apparatus of classical elasticity theory in the determination of stress and strain near the margin of a crack in work [35] is not justifiable, since, in using this apparatus, all distances — even those which are assumed to be small — must be large in comparison with the interatomic distances. In addition, it is necessary to consider that the cohesive forces act not only within a body but also on sections of the surface of the crack. As will be shown in detail below, consideration of this circumstance gives a pointed shape to the ends of the crack rather than a rounded shape, with no infinite concentration of stresses at the ends of the crack. Thus, the distributions of stresses and shears near the edge of the crack surface differ substantially from the corresponding distributions obtained in accordance with the solutions proposed by Inglis [1] and Sack [20], in which the surface of the crack was assumed to be free of stress. Let us note that the drop observed in the curve σ_y (2v) with increasing v is considered in work [35] as occurring very slowly, far more slowly than the natural rate of drop in the intensity of the cohesive forces.

Ya.I. Frankel' [5] dealt with the problem of a crack of infinite length which passes longitudinally along a thin band. The use of the approximate theory of thin beams, which is useless for investigation of the form of a crack near its ends, did not allow him to obtain adequate results. We may note in passing that the critique of Griffiths' theory contained in this work by Ya.I. Frankel' likewise cannot be considered correct to any substantial degree. Ya.I. Frankel' questions Griffiths' statement that equilibrium is unstable in the case of a rectilinear crack in a uniform tensile-stress field as considered by the latter, relating this instability to the incorrect assumption on the part of Griffiths as to the form of the ends of the crack. This was wrong: the structure assumed for the crack at its ends has no bearing on the stability or instability of the crack equilibrium. As will be shown below, crack instability in a homogeneous field also takes place on considering the smooth union of the cracks at their ends; it corresponds fully to the essence of the matter. Ya.I. Frankel's conclusion that, in addition to an unstable state, a stable equilibrium state also exists in this case was brought about by his incorrect substitution of another stressed state for the homogeneous stressed state.*

The work by A.R. Rzhanitsyn [37] made an attempt to solve the problem of a circular crack in a body subject to a uniform tensile stress with consideration of molecular cohesive forces distributed along the surfaces of the crack and a smooth union of the crack at its edge. Unfortunately, the use of inadequate methods based on the averaging of stresses and strains made it impossible for the author to obtain the correct equilibrium conditions for the crack.

The idea first introduced by S.A. Khristianovich [38] is of basic importance for understanding the structure of cracks in the vicinity of the ends. In connection with the theory of the so-called hydraulic fracture of an oil-bearing geological stratum, S.A. Khristianovich dealt with an isolated crack in an infinite body compressed at infinity by a constant hydrostatic stress; the crack was supported by the uniformly distributed pressure of a fluid enclosed within the crack. The problem was studied in a quasistatic formulation. In its solution, S.A. Khristianovich was balked by the indeterminate length of the crack. However, he drew attention to the following circumstance. If we assume that the liquid fills the crack completely, the fracture stress at the end of the crack is always infinitely large, whatever the size of the crack. But if we assume that the liquid does not fill the crack completely, so that there is a free section of the surface of the crack which is not wetted by the liquid, the fracture stresses at the ends of the crack will be finite at one exceptional value of the crack length. For this crack length (and only for this length) it was found that the opposing faces of the crack unite smoothly at its ends. S.A. Khristianovich advanced the hypothesis of finite stress, or, what is the same thing, smooth uniting of the opposing faces of the crack at its ends, as a basic condition determining the size of the crack. Use of this hypothesis has made it possible to solve a number of problems in the formation and growth of cracks in rocks [38-43]. However, none of these works considered the molecular cohesive forces directly. In dealing with cracks in rock masses, it is quite permissible to neglect the cohesive forces, as was shown by the evaluations, since the pressure of the surrounding rock mass is manifested here much more strongly than the molecular cohesive force, especially if we consider the naturally broken-up nature of the rocks. Under other conditions (in particular, in many cases where laboratory models of rock massifs are used), the cohesive forces play an important role and their consideration is of substantial importance for analysis of equilibrium conditions and the development of cracks.

In connection with this research, we should note the very interesting earlier work of Westergaard [44] (see also [13]). This work, on the basis of an analogy with the contact problem noted by the author, affirms the absence of stress concentration at the end of a crack in a concrete-like brittle material. Reference [44] also gives formulas which correctly describe the stresses and strains in the vicinity of the ends of the equilibrium cracks formed in brittle fracture in the absence of cohesive forces. However, Westergaard did not relate the finite-stress condition to the determination of the length of the crack, which he assumed to be given.

The studies by Irwin [45, 46] (see also [47, 48, 49, 33]) established an important formula which related the rate of elastic energy liberation to the coefficient of stress intensity in the vicinity of the ends of the crack in the problem of the classical elasticity theory. The rate of elastic-energy liberation and the fracture conditions for several new cases of loading and crack position were determined on the basis of this formula [47, 50, 32, 51, 52].

Beginning with Griffiths, the majority of theoretical research has dealt with problems of one type, in which the equilibrium state in which the intensity of the cohesive forces at the edge of the craft is maximal is unstable and the condition necessary for the development of a crack to begin was identical with the condition necessary for complete fracture to begin. Consequently, some works identified the condition for initiation of crack development with the condition of rapid crack propogation and fracture for all cracks. Generally speaking, this is not so; actually, cracks can be stable, so that the start of the crack development is not at all necessarily associated with fracture of the body. We must not treat this matter as though stable cracks were a rarity not encountered in practice and difficult to generate experimentally. As the experimental research carried out by various authors, beginning with I.V. Obreimov [8], has shown, in many cases the development of cracks proceeds stably during considerable portions of the fracture process. Thus, Wells [30] obtained cracks in steel plates which were stable over a certain tensile-stress range to the combined action of external tensile stresses and internal stresses set up by welded seams. Roesler [53] and Benbow [54] investigated stable conical cracks in glass and quartz. Benbow and Roesler [9] obtained stable cracks by inserting wedges in strips of organic glass. Recently, Romualdi and Sanders [52] obtained cracks that were stable within definite stress ranges by elongating a plate reinforced by riveted-on stiffening ribs. References to other research in which stable cracks were obtained and studied can be found in the monograph by B.A. Drozdovskiy and Ya.B. Fridman [55]. All these works definitely confirm the feasibility of applying the concept of brittle and quasibrittle fracture to stable cracks.

Consideration of stable cracks greatly broadens problem formulation in the theory of equilibrium cracks. Actually, only determination of the load at which the crack begins to widen is of interest for unstable cracks, since the process of crack development before this stress is reached becomes dynamic. For stable cracks, there also arises the problem of investigating the quasistatic development of cracks with varying loads.

In connection with the foregoing considerations, References [56-61] clarify and supplement formulation of problems in the theory of equilibrium cracks formed in brittle fracture. These works proposed a new approach to the problems of crack theory based on the general presentation of the problem of elastic equilibrium in a body containing cracks as formulated in [40]. The material which follows is based on this approach, so that we shall not dwell on its characteristics here. A number of new problems of the theory of cracks have been formulated and solved on the basis of the proposed approach.

III. Structure of Ends of Equilibrium Crack in a Brittle Body

1. Stresses and strains in the vicinity of the end of an arbitrary normal-shearing-fracture surface. As was shown earlier, it is possible to construct a formal solution to the differential equa-

tions of elasticity theory which satisfies the boundary conditions corresponding to the load applied to the body by arbitrary assignment of the shearing-fracture surface. This section is a study of the behavior of solutions of the elasticity-theory equations in the vicinity of the edge of a shearing-fracture surface. For simplicity of discussion, we shall limit ourselves here to normal-shearing-fracture surfaces which are sections of a surface bounded by closed contours.

Let us take a neighborhood in the vicinity of an arbitrary point O on the boundary of such a surface whose characteristic dimension is small in comparison with the radius of curvature of the boundary at point O. The deformation in this region can be assumed two-dimensional and to correspond to an infinite rectilinear slit in an infinite body being acted upon by a certain system of symmetrical loads (Fig. 6; the plane of deformation is the plane normal to the contour of the fracture surface at point O, and the line of the slit is the intersection of this plane with the fracture surface). Loads can be applied to the surfaces of the slit and within the body; loads applied to the surfaces of the slit may be regarded as normal without loss of generality in subsequent analysis. Let us consider this configuration in greater detail.

The field of stresses and shears can be presented as the sum of two fields (Fig. 6), the first of which corresponds to a continuous body acted upon by a stress applied from within the body, while the second corresponds to a body containing a slit and acted upon by symmetrical loads applied only to the surfaces of the slit. The shape of the deformed surface of the slit is determined by the second stressed state, since the normal shears at the site of the slit in the first stressed state are, according to symmetry, zero.*

Analysis of the first stressed state is carried out by the usual methods of elasticity theory and is of no basic interest; we will consider this stressed state to be known. Let us assume that the line of the slit corresponds to the positive x-semiaxis; the normal

Fig. 6.

stresses g(x) applied to the surface of the slit in the second stressed state are equal to the difference between the stresses applied to the surface of the slit in the resultant field G(x) and the stresses at the slit p(x) which correspond to the first stressed state.

Using Muskhelishvili's method [18] for analysis of the second stressed state, we have relationships determining the stresses and shears in the form

$$\sigma_x^{(2)} + \sigma_y^{(2)} = 4\,\mathrm{Re}\,\Phi(z) \tag{3.1}$$

$$\sigma_y^{(2)} - i\sigma_{xy}^{(2)} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)}$$

$$2\mu\,(u^{(2)} + iv^{(2)}) = \varkappa\varphi(z) - \omega(\bar{z}) - (z - \bar{z})\overline{\Phi(z)} \tag{3.2}$$

$$\varkappa = 3 - 4\nu \tag{3.3}$$

Here $z = x + iy$, $\sigma_x^{(2)}$, $\sigma_y^{(2)}$ and $\sigma_{xy}^{(2)}$ are the components of the stress tensor for the second stressed state; $u^{(2)}$ and $v^{(2)}$ are the shear components along the <u>x</u> and <u>y</u> axes corresponding to the second stressed state; $\mu = E/2(1 + \nu)$ is the shear modulus, E is Young's modulus, and ν is Poisson's ration. The analytical functions φ, ω, Φ, and Ω are expressed by the formulas

$$\Phi(z) = \Omega(z) = \varphi'(z) = \omega'(z) = \frac{1}{2\pi i\sqrt{z}}\int_0^\infty \frac{\sqrt{t}\,g(t)\,dt}{t - z} \tag{3.4}$$

$$\varphi(z) = \omega(z) = \frac{1}{2\pi i}\int_0^\infty g(t)\ln\frac{\sqrt{t}+\sqrt{z}}{\sqrt{t}-\sqrt{z}}\,dt \tag{3.5}$$

At the slit ($x \geq 0$, $y = 0$) and its extension ($x \leq 0$, $y = 0$), the expressions

$$\sigma_x^{(2)} = \sigma_y^{(2)} = 2\,\mathrm{Re}\,\Phi(z),\ \sigma_{xy}^{(2)} = 0, \qquad v^{(2)} = \frac{4(1-\nu^2)}{E}\,\mathrm{Im}\,\varphi(z) \tag{3.6}$$

are satisfied.

From this and from the known formulas for the limiting values of a Cauchy integral at the ends of the contour [19], an expression is obtained for the normal tensile stresses in the vicinity of the end of the slit on its extension:

$$\sigma_y^{(2)} = -\frac{1}{\pi\sqrt{s_1}}\int_0^\infty \frac{g(t)\,dt}{\sqrt{t}} + g(0) + O(\sqrt{s_1}) \tag{3.7}$$

where s_1 is the short distance from the point being considered to the end of the slit. Similarly, in order to determine the normal displacement for points on the slit surfaces in the vicinity of its end, we obtain

$$v^{(2)} = \pm\frac{4(1-\nu^2)}{\pi E}\sqrt{s_2}\int_0^\infty \frac{g(t)\,dt}{\sqrt{t}} + O(s_2^{3/2}) \tag{3.8}$$

where s_2 is the distance from the point on the slit surface being considered to its end, while the plus and minus signs correspond to the upper and lower faces of the slit.

The research which has been conducted has completely clarified the distribution of normal tensile stresses and normal shears in the vicinity of the boundary of an arbitrary normal fracture surface. Specifically, the formulas

$$\sigma_y = \frac{N}{\sqrt{s_1}} + G(0) + O(\sqrt{s_1}), \quad v = \mp\frac{4(1-\nu^2)N\sqrt{s_2}}{E} + O(s_2^{3/2}) \tag{3.9}$$

follow directly from Expressions (3.7) and (3.8). Here, σ_y is the tensile stress at a point on the body lying at a short distance s_1

from the boundary of the fracture surface and on a plane contiguous to the contour of the fracture surface passed at point O; N is the "coefficient of stress intensity", whose value depends on the loads acting, the configuration of the body and its fracture surface, and the coordinates of the point O on the contour being considered; G(O) is the magnitude of the normal stress applied to the fracture surface at the point on the contour of this surface being considered (Fig. 6); s_2 is the short distance between the point on the fracture surface and its contour. Generally speaking, there are three possibilities, depending on the sign of N.

If $N > 0$, an infinite tensile stress acts at point O on the boundary of the fracture surface. The shape of the deformed fracture surface and the distribution of the normal stresses σ_y in the vicinity of point O have the form shown in Fig. 7a.

If $N < 0$, an infinite compressive stress acts at point O on the boundary; the shape of the deformed fracture surface and the distribution of stresses σ_y in the vicinity of point O have the form shown in Fig. 7b. As may be seen, in this case the opposing faces of the crack enter one another and, as it were, merge; it is obvious that this case is physically impossible.

Finally, if $N = 0$, the stress acting in the vicinity of the boundary is limited and as we approach point O, it tends toward the normal stress applied to the surface at this spot on the boundary, so that there is a continuity of the stresses σ_y at the boundary and a smooth union of the opposing faces of the fracture surface at its boundary (Fig. 7c).

Research on the distribution of stresses and strains in the vicinity of the edge of a normal fracture surface was begun by Westergaard [44, 13] and Sneddon [14, 15] and subsequently continued by the author [40], Williams [17], and Irwin [45-47]. Due to the nature of the stressed states considered in works [14, 15] and [45-47], results were obtained which had a bearing only on the case $N > 0$.

Fig. 7.

2. Stresses and strains in the vicinity of the edge of an equilibrium crack. The results obtained in the preceding section pertain to an arbitrary normal slip-fracture surface. Let us prove that, for an equilibrium crack, $N = 0$ at all points on its boundary.

Let us consider the possible state of an elastic system which differs from the actual equilibrium state only by a certain variation in the form of the contour of the crack in a small area around an arbitrary point O on it (Fig. 8). The new contour is a certain curve surrounding point O in the plane of the crack. This curve is in contact with the previous boundary of the crack. This curve is in contact with the previous boundary of the crack at points A and D near O; at all other points, the contours of all cracks remain unchanged. Because of the proximity of the points of contact A and B to point O, the initial contour of the crack contour can be assumed rectilinear on segment AB. According to the foregoing, the distribution of normal shears for points on the new crack surface and the distribution of tensile stresses at these points before the formation of the new crack surface have, to within small quantities, the following form:

$$v = \pm \frac{4(1-\nu^2)N}{E}\sqrt{h-y}, \quad \sigma_y = \frac{N}{\sqrt{y}} \qquad (3.10)$$

Fig. 8.

Here N is the coefficient of stress intensity at point O.

The energy liberated in the formation of the new crack surface, which is equal to the work required to close this new surface is obviously equal to

$$\delta A = \frac{1}{2}\,2\int_{\delta S} \sigma_y |v|\, dS = \frac{4(1-\nu^2)N^2}{E}\int_a^b dx \int_0^h \sqrt{\frac{h-y}{y}}\, dy =$$
$$= \frac{2(1-\nu^2)\pi N^2}{E}\int_a^b h\,dx = \frac{2(1-\nu^2)\pi N^2 \delta S}{E} \qquad (3.11)$$

where δS is the area of the projection of the new crack surface onto its plane.

It follows from the equilibrium conditions of the crack that δA should revert to zero, from which and (3.11) it follows that

$$N = 0.$$

Thus, a very important statement characterizing the structure of the cracks in the vicinity of their contours is valid.

1. The tensile stresses at the boundary of the crack are finite.

2. The opposite banks of a crack unite smoothly at its boundary.

Thus it has been shown that, in contrast to Griffiths' ideas, the form of the crack in the vicinity of the edge is as that depicted in Fig. 4. Since the only forces which act upon the surface of the crack in the vicinity of its boundary are cohesive forces, it follows from Eq. (3.9) that the tensile stress at the boundary of the crack equals the cohesive force intensity at the boundary.

In particular, if there are no cohesive forces, the tensile stress at the boundary of the crack will equal 0.

The condition of finite stress and smooth union of the opposing faces at the ends of a crack was first suggested in hypothetical form by S.A. Khristianovich [38] as the basic condition which determines the position of the end of the crack. The proof given above for this condition follows basically from [60]. Formula (3.11) for plane deformation was first indicated without relation to finite stress and smooth union in the work of Irwin [45,46] (see also the survey by Irwin [47] and the paper by Bueckner [33]). The earlier work by Westergaard [44] affirmed the absence of stress concentration at the end of the crack in a brittle material of the concerete type, although the finite-stress condition was not associated in this work with determination of the size of the crack.

Fig. 9.

We are considering here cracks involved in normal fracture solely for simplicity of description. The entire foregoing discussion and, in particular, the demonstration of the finite magnitude of the stresses at the end of a crack can be extended without any substantial change to the general case in which the surfaces of the crack undergo fracture and are subject to tangential slip components.

3. Determination of the boundaries of equilibrium cracks. The condition of finite stress and smooth union of a crack at its boundary makes it possible, for a given system of forces acting on a body, to formulate the problem of the theory of equilibrium cracks. This problem consists in the following. For a given arrangement of initial cracks and a given system of forces acting on a body, it

is necessary to find the stress, deformation, and crack boundaries in the elastic body under consideration so as to satisfy the differential equations of equilibrium and the boundary conditions and to ensure finite stresses and smooth union of the opposing faces at the boundaries of the crack.

Let us analyze the solution of this problem on an elementary model of an isolated rectilinear crack in an infinite elastic solid which is compressed at infinity by a nondirectional stress q. The crack is subject to the concentrated forces T, which are applied at opposing points on its surface (Fig. 9).

We can use the method of N.I. Muskhelishvili [18] to obtain a solution for the equilibrium equations which satisfy the boundary conditions for an arbitrary crack length 2l. In this case the stresses and shears are expressed by Formulas (3.1)-(3.3), with

$$\Phi(z) = \frac{2\zeta^2}{l(\zeta^2 - 1)}\left\{\frac{P}{\pi(\zeta^2 + 1)} - \frac{ql(\zeta^2 + 1)}{4\zeta^2}\right\} \qquad (3.12)$$

$$z = \frac{l}{2}\left(\zeta + \frac{1}{\zeta}\right)$$

As may be seen, the equilibrium equations and boundary conditions do not determine the length of the crack. The distribution of stresses σ_y over the extent of the crack and the normal shears v for points on the surface of the crack in the vicinity of its end is given in the form

$$\sigma_y = \left(\frac{P}{\pi l} - q\right)\sqrt{\frac{l}{8s_1}} + O(1) \qquad (3.13)$$

$$v = \pm\frac{(1 - \nu^2)}{E}\left(\frac{P}{\pi l} - q\right)\sqrt{8s_2 l} + O(s_2^{3/2})$$

The finite stresses and smooth union of the crack at its ends are ensured simultaneously by the condition

$$l = \frac{P}{\pi q} \qquad (3.14)$$

which also determines the size of the crack for given stresses T and q.

Let us now try to determine the length 2l of an isolated rectilinear crack in an infinite body under tension at infinity by a uniform stress P_0 in the direction perpendicular to the crack. If we assume that the surface of the crack is free from stress, it is not difficult to show that the tensile stress over the length of the crack in the vicinity of its end depends on the distance s_1 in the following manner:

$$\sigma_y = \frac{p_0 \sqrt{l}}{\sqrt{2s_1}} \tag{3.15}$$

From this it follows that, at any $\underline{l}$ σ_y at the end of the crack will not be finite and there will be no equilibrium cracks. This paradoxical result is explained by the fact that we have not taken into consideration the molecular cohesive forces acting in the vicinity of the boundaries of the crack on its surfaces, and have thus incompletely characterized the loads acting on the body.

Consideration of cohesive forces and the final formulation of the problem of the theory of equilibrium cracks formed in brittle fracture are dealt with in the following Section.

IV. Basic Hypotheses and General Formulation of the Problem of Equilibrium Cracks

1. Cohesive forces. Terminal and interior regions. Basic hypotheses. In order to construct an adequate theory of the cracks formed in brittle fracture, it is necessary to supplement our model of the brittle body by considering the molecular cohesive forces acting on the surfaces of a crack in the vicinity of its end. As we know, the intensity of cohesive forces varies greatly as a function of distance. Thus, for an ideal crystal, the intensity $\underline{f}$ of the cohesive forces acting between two atomic planes at a distance $\underline{y}$ from one another equals 0 when $\underline{y}$ equals the normal interatomic distance $\underline{b}$. When $\underline{y}$ increases to a magnitude of the order of one-and-one-half times $\underline{b}$, the intensity $\underline{f}$ increases, reaching a very high maxi-

mum value $f_m \approx \sqrt{ET_0/b} \approx E/10$, and then rapidly decreases with increasing y (Fig. 10).

Here E is Young's modulus and T_0 is the surface tension, which is related to $f(y)$ by the expression

$$2T_0 = \int_0^\infty f(y)\,dy \tag{4.1}$$

Fig. 10.

The maximum intensity f_m defines the theoretical strength, i.e., the strength which the solid would have if it were an ideal crystal.

The actual strength of a solid is generally several orders of magnitude lower because of the presence of defects in the crystal structure.

For an amorphous body, the relationship of the intensity of cohesive forces to distance has the same qualitative character.

At the present time, the data which confirm the character of the relationship between the intensity of cohesive forces and distance stated above reduce to the following. It has long been known that the strength of thin filaments considerably exceeds the strength of large specimens produced from the same material [62, 63]. Recent experiments have brought to light the exceptionally high strength of thread-like crystals of certain metals; this strength approximated the theoretical values [63]. We may assume that this phenomenon is associated with a comparatively small number of structural defects in thin filaments and whisker crystals. Further, numerous direct measurements have recently been conducted on the intensity of molecular cohesive forces in glass and quartz [64-66]. Special mention should be made of the highly elegant method of the types of measurement based on the use of micro-valances employing feedback, which were proposed and used by B.V. Deryagin and I.I. Abrikosova [64, 65]. However, this direct measurement is for determination of the distance *y*, which is very large in comparison with the normal interatomic distance,

Fig. 11.

and thus determines only the end of the descending arm of the curve f(y). Ye.M. Lifshits [64] developed a macroscopic theory for the cohesive forces at such distances; this theory has been well confirmed by the results of the measurements mentioned above. At distances of the order of several normal interatomic distances, the function f(y) is inaccessible at the present time to any rigorous quantitative theory or to direct experimental determination. An account of attempts at mathematical evaluation of the function f(y) for such distances and the theoretical strength can be found in [67, 63, 68].

The distance between the opposing faces of a crack varies from magnitudes of the order of interatomic distances in the vicinity of the crack contour to occasionally rather high values remote from the boundary. Consequently, it is natural to divide the surface of the crack into two parts (Fig. 11). In the first part, *the interior region of the crack*, the opposing faces of the crack are far apart, so that their interaction is negligibly small and the surface of the crack can be assumed to be free of stresses caused by the interaction of the opposing faces. In the second section, which adjoins the contour of the crack and is referred to as the *terminal region of the crack*, the opposing faces of the crack draw near one another, so that the action of molecular cohesive forces on this section of the surface is of considerable intensity.

The boundary between the terminal and interior regions of the crack surface is, of course, to a certain extent arbitrary. For very small cracks, the interior region of the crack surface may not exist at all.

Since the distribution of cohesive forces along the surface of the terminal region of the crack is not known beforehand, a substantial part of the loads applied to the body is not known either. Consequently, it is impossible to solve the problem of cracks directly in the form in which it is stated in Section III. The following is possible in principle for solving the problem of cracks. The distance between the opposing crack faces at each point on its surface is determined as a function of the unknown distribution of cohesive forces along the surface. Assuming an assigned relationship f(y) expressing the intensity of cohesive forces as a function of distance, we may find from it an expression determining the distribution of cohesive forces along the surface of the crack.

This approach to the problem of cracks cannot be carried out in practice. First of all, the function f(y) is not known to a sufficient extent for any real material. Even if this function were known, the problem would reduce to a very complex nonlinear integral equation whose effective solution presents great difficulty even in the simplest cases.*

Attempts have been made to assign a definite form to the distribution of cohesive forces along the surfaces of the crack, but these attempts cannot be considered sufficiently substantiated.

For rather large cracks, investigation of which is of basic interest, the difficulty associated with the lack of information on the distribution of cohesive forces along their surfaces can be avoided by not making any concrete hypotheses about this distribution. More precisely, the general properties considered above for cohesive forces as a function of distance make it possible to formulate two basic hypotheses which substantially simplify further analysis and make it possible in the final analysis to eliminate the cohesive forces completely from consideration of the loads acting on the body in determining the contours of the cracks.

First hypothesis. The width d of the terminal region of the crack is small in comparison with the size of the entire crack.

The possibility of adopting this first hypothesis is a result of the rapid decrease in cohesive forces when the distance between the opposing faces of the crack is increased.

It is understood that there are microcracks to which this hypothesis is inapplicable. However, since the width d of the terminal region is very small, the first hypothesis is correct for very small cracks and is known to be correct for all real macrocracks. All the same, the width d is assumed to be sufficiently great in comparison with microscopic dimensions (for example, in comparison with the lattice constant of a crystalline body) that the metheds of continuum mechanics can be used for distances of the order of d.

Second hypothesis. The form of the normal section of the surface of the crack in the terminal region (and, consequently, the local distribution of cohesive forces along the surface of the crack) does not depend on the loads acting on the crack and is always the same for a given material under given conditions (temperature, composition and pressure of the surrounding atmosphere, etc.). (By normal section, we mean here a section cut by a plane normal to the contour of the crack.)

According to the second hypothesis, when a crack widens, the terminal region in the vicinity of a given point shifts progressively, as it were, to another place but the shape of its normal section remains unchanged.

The second hypothesis is applicable only for those points on the contour of the crack where the maximum possible intensity of cohesive forces is reached, so that any increase, no matter how small, in the load applied to the body at this point causes the crack to widen.

Equilibrium cracks whose contours have at least one such point are naturally called mobile-equilibrium cracks in contrast to stationary-equilibrium cracks, which do not possess this property and, consequently, do not widen on an infinitesimally small increase in load.

Thus, the second hypothesis and all conclusions derived from it are applicable to reversible cracks, as well as to the irreversible equilibrium cracks which are formed in primary fracture of a brittle body while the stress is increasing. They are applicable to irreversible cracks formed by a decrease in the load on equilibrium cracks that existed under some large load, and to artificial notches which do not widen subsequent to their formation.

The possibility of adopting this second hypothesis is associated with the fact that the maximum intensity of cohesive forces is very large and exceeds by several orders the stresses which would arise in a solid body without cracks which is subject to the same load. It is therefore possible to neglect changes in stress occurring in the terminal region as a result of a change in load and, consequently, the corresponding changes in the form of the normal sections of the terminal region.

The hypotheses formulated are a synthesis of the results of a qualitative analysis of the phenomena of brittle fracture carried out by a number of investigators, beginning with Griffiths. They are the only hypotheses dealing with cohesive forces and form the basis for the theory given below. They are formulated in explicit form in [56, 57].

2. The coefficient of cohesion. It is assumed that the body being considered is linearly-elastic to the point of failure, so that the field of the elastic elements in the body containing the cracks can be represented as the sum of two fields; the field calculated without considering cohesive forces and a field corresponding to the action of the cohesive forces alone. The quantity N which occurs in formula (3.15) and, Q.E.D., equals zero can therefore be represented in the form $N = N_0 + N_m$, where the coefficient of stress intensity N_0 corresponds to the loads acting on the body and the same crack configuration but without considering the cohesive forces, while the coefficient of stress intensity N_m corresponds to the same crack configurations and the cohesive forces taken alone.

By virtue of the first hypothesis, the width d of the terminal region in which the cohesive forces act is small in comparison with the dimensions of the cracks as a whole and, in particular, in comparison with the radius of curvature of the crack contour at the point under consideration. Consequently, it is possible in determining the values of N_m, to assume that the field corresponds to the configuration of an infinite body with a semiinfinite slit, as considered in Section III, Paragraph 1, to whose surface symmetrical normal stresses are applied. From this and from (3.7) it follows that

$$N_m = -\frac{1}{\pi}\int_0^\infty \frac{G(t)\,dt}{\sqrt{t}} = -\frac{1}{\pi}\int_0^d \frac{G(t)\,dt}{\sqrt{t}} \tag{4.2}$$

Here G(t) is the distribution of cohesive forces differing from zero only in the terminal region $0 \leq t \leq d$.

By virtue of this second hypothesis, the distribution of cohesive forces and the width d of the terminal region at those points on the contour of the crack where the intensity of cohesive forces is at a maximum are independent of the load applied, so that the integral on the right-hand side of Eq. (4.2) is a constant characteristic of the given material under given conditions. This constant is designated K:

$$K = \int_0^d \frac{G(t)\,dt}{\sqrt{t}} \tag{4.3}$$

and is called the modulus of cohesion, since this quantity characterizes the crack-development resistance of the material due to the cohesive forces. As will be shown later, the quantity K is the only characteristie of the cohesive forces which takes part in formulation of the problem of cracks.

The dimensions of the modulus of cohesion are

$$[K] = [F][L]^{-3/2} = [M][L]^{-1/2}[T]^{-2} \tag{4.4}$$

Here [F] represents the dimensions of force, [L] length, [M] mass, and [T] time. Constants with similar dimensional formulas are encountered in the contact problem of elasticity theory [71, 72, 73]. It is not coincidental that a strong correlation exists between contact problems and problems in the theory of cracks generated in brittle fracture, as was apparently first noted in works by Westergaard [44, 13].

3. Boundary conditions at outline of equilibrium crack. For points on the boundary of an equilibrium crack at which the maximum intensity of cohesive forces is reached, so that the second hypothesis is applicable, Formula (4.2) is represented in the form

$$N_m = -\frac{1}{\pi} K \qquad (4.5)$$

From this, and since N = 0, we obtain

$$N_0 = \frac{1}{\pi} K \qquad (4.6)$$

It is also possible to formulate the boundary condition at points on the boundary of an equilibrium crack at which the intensity of cohesive forces is at a maximum in the following fashion. As we approach these points, the normal tensile stress σ_y at points in the body lying in the plane of the crack, as calculated without considering the cohesive forces, tends to infinity in accordance with the expression

$$\sigma_y = \frac{K}{\pi \sqrt{s}} + O(1) \qquad (4.7)$$

where s is a short distance from the contour point being considered.

Satisfaction of (4.6) for at least one point on the contour will be the condition under which the crack reaches a state of mobile equilibrium.

It should be specially emphasized that generally speaking, a crack's reaching a state of mobile equilibrium should not be associated with the beginning of its rapid unstable development and, even less with complete failure of the body. A mobile-equilibrium crack can be either stable or unstable. Only in the case of unstable mobile equilibrium will Eq. (4.6) be the condition for the beginning of rapid crack development. However, even in this case, complete failure of the body is not obligatory: the crack may shift from an unstable equilibrium to another, stable state. Numerous examples illustrating various possibilities will be considered in the next section.

If the crack is irreversible and if there are points on its boundary where the intensity of cohesive forces is less than the maximum possible value*, the second hypothesis is inapplicable at such points. The cohesive forces acting in the terminal region of the crack surface in the vicinity of such points are smaller than the cohesive forces acting in the terminal region in the vicinity of points of the type considered above. Consequently, it follows from (4.2) that $-N_m < K/\pi$ and, since $N_0 = -N_m$, that for such points

$$N_0 < \frac{K}{\pi} \tag{4.8}$$

With increasing load, the cohesive forces in the terminal region increase, ensuring finite stress and a smooth union at the boundary of the crack. However, the crack will not widen at this point on the boundary until the cohesive forces reach their maximum intensity, so that the second hypothesis becomes applicable and Condition (4.6) is satisfied.

Fig. 12. 1) G.

In determining the form of the boundaries of equilibrium cracks, Conditions (4.6) and (4.8) make it possible to eliminate the cohesive forces altogether from consideration of the loads acting on the body and to limit them to the resultant integral characteristic, the modulus of cohesion. Special evaluations have shown [57, 58] that the effect of molecular cohesive forces on the stress and shear fields is essential only in the vicinity of the terminal region of a crack having a size of the order of the width d of the terminal region. Thus, the cohesive forces determine the structure of the crack in the vicinity of its ends and, only through their integral characteristic K, the form of the boundaries of the crack.

4. Basic problems of the theory of equilibrium cracks. In its most general form, the basic problem of the theory of equilibrium cracks may be stated in the following fashion. We are given a certain system of initial cracks and a process for loading the body, i.e., a system of loads acting on the body and depending on a single monotonically increasing parameter λ. For the initial state, the value of λ can be assumed to be zero. It is necessary to determine the form of the surface of the crack as well as to find the distribution of stresses and shears in the body which corresponds to $\lambda > 0$. It is assumed that the load varies quite slowly so that dynamic effects need not be considered.

When the body, the load, and the initial cracks are symmetrical, thus making it possible for a system of plane cracks to develop, and the tensile stress increases monotonically with increasing λ, the configuration of the cracks in the body is determined solely by the current load, and not by the cumulative effects of previous loads, as in the general case. Here, the problem of the theory of equilibrium cracks is formulated in the following fashion (we shall call this Problem A). In a body bounded by a surface Σ, the boundaries of an initial system of surface cracks G_0 are assigned (Fig. 12; the plane of the drawing is the plane of the cracks). It is necessary to find the field of the elastic elements and the boundaries G of the system of surface which incurs the boundary G_0 (which may partially coincide with it) corresponding to the given load, i.e., the given value of λ.

Mathematically, the problem formulated reduces to the following. It is necessary to construct a solution for the differential equilibrium equations of elasticity theory in a region bounded by plane slits with the contours G and by the boundary Σ of the body, with boundary conditions corresponding to the given load. In this case, the boundaries G should be defined so that Conditon (4.6) is satisfied at points on these boundaries not lying on G_0, while Eq. (4.8) will be fulfilled for points of G lying on G_0.

If the cracks are reversible or if the loads applied are quite large, so that the boundaries G do not coincide with G_0 at even one point, the form of the initial boundaries is of no significance. It is therefore possible, without assigning the initial cracks, to set up the problem of determining the boundaries G of a given configuration of equilibrium cracks directly in such a way that Condition (4.6) is satisfied at each point of G. Here it is assumed that the initial cracks are such that they ensure the formation of the given crack configuration on an increase in load. In this form, the problem is called problem B.

It might be found that no solution exists for any of the problems posed here. Physically, however, this circumstance has totally different interpretations for problems A and B. If there is no solution to problem A, this means that the load applied exceeds the fracture stress, so that failure of the body intervenes when it is applied. The limiting value of the parameter λ, below which a solution exists for problem A, corresponds to the failure stress. The determination of the failure stress for a given original crack configuration and a given system of loads is an important problem of the theory of cracks. The nonexistence of a solution to problem B means that whatever the original cracks within the given configuration, they do not increase in size under the action of the given load, indicating that the load applied is too small. In such cases, we may provisionally say that mobile-equilibrium cracks are not formed at the given load.

5. The energy method of deriving the boundary condition at the contour of an equilibrium crack. Until now, the molecular cohesive forces have been considered as external forces applied to the surface of the body. This was necessary in order that we might study the structure of the crack in the vicinity of its ends. If we wish only to obtain the boundary condition, we can use another approach, considering the cohesive forces as forces within the system. On the basis of this approach, which occurred to Griffiths [3, 4], we can show the relationship between the modulus of cohesion and the other characteristics of the material.

As before, let us assume that there is a certain configuration of equilibrium cracks in a brittle body. As in Section III, Paragraph 2, we will turn to a possible state of the elastic system which differs from the actual state only by a change in the boundary of the crack in the vicinity of a certain point O (Fig. 8). However, in a departure from Section III, Paragraph 2, we assume that the characteristic dimension of the new region of the crack surface is large in comparison to the width $\underline{d}$ of the terminal region, although it is small as compared with the size of the entire crack; according to the first hypothesis presented in Section IV, Paragraph 1, such an assumption is permissible. In this hypothesis, the cohesive forces can be considered simply as forces of surface tension. In order to overcome these forces, some work is expended in increasing the area of the crack. The effect of the cohesive forces on the field of elastic stresses and strains can be disregarded, since it is substantial only in the vicinity of the end of a crack, which has a size on the order of the width of the terminal region.

The work δA expended in the transition from the actual state to the possible state is equal to the difference between the corresponding surface-energy increment δU and the elastic energy δW liberated:

$$\delta A = \delta U - \delta W \tag{4.9}$$

In order for the actual state of the elastic system to be an equilibrium state it is necessary that δA revert to zero, so that

$$\delta U = \delta W \tag{4.10}$$

In quite the same way as in Section III, Paragraph 2, we obtain an expression for δW:

$$\delta W = \frac{2(1-\nu^2)\pi N_0^2 \delta S}{E} \tag{4.11}$$

Here N_0 is the coefficient of stress intensity at the point 0, calculated without consideration of the cohesive forces. Formula (4.11) was established by Irwin [45-47] in somewhat different form.

If the form of the terminal region of a crack in the vicinity of a given point on its contour corresponds to the maximum intensity of cohesive forces, according to what has preceded, the terminal region will move when a new surface is formed by the crack and will not be deformed, so that the work opposing the cohesive forces in the formation of a unit of new surface area is constant and equals the surface tension T_0. Consequently, $\delta U = 2T_0\delta S$ (the 2 is made necessary by the formation of two crack surfaces in fracture). Hence, from (4.10) and (4.11) we obtain

$$N_0 = \sqrt{\frac{ET_0}{\pi(1-\nu^2)}} \tag{4.12}$$

Comparing (4.12) and (4.6), we obtain an expression which relates the modulus of cohesion K determined independently in accordance with (4.3) to the surface tension T_0 and the elastic constants E and ν of the material

$$K^2 = \frac{\pi E T_0}{1-\nu^2} \tag{4.13}$$

6. Experimental confirmations of the theory of brittle fracture. Quasibrittle fracture. Beginning with Griffiths [3, 4], various investigators attempted to verify experimentally the theory of brittle fracture. We do not propose to make any detailed analysis of all these works here, but will devote ourselves only to a few of the most characteristic, referring the reader to specialized papers for details and discussions of the numerous other studies [62, 55, 74-78].

In Griffiths' work [3], the following experiments are described and their results given. Cracks of various lengths $2l$ were formed on spherical glass flasks and cylindrical tubes whose diameters D were sufficiently large so that a special test demonstrated the absence of any effect caused by the diameter of the vessel on the results of the experiments. The tubes and flasks were then annealed in order to relieve the internal stresses formed during the generation of the cracks and then were loaded internally by hydraulic pressure until they failed. The failure stress p_0 corresponding to each crack length $2l$ was measured in the vessels.

In accordance with the theory presented above, it was found that the failure stress p_0 at which a given crack became an unstable mobile-equilibrium crack depended only on the length of the crack $2l$ and the modulus of cohesion K, so that dimensional analysis [79] showed that $p_0 = \alpha K \sqrt{l}$, where α is a nondimensional constant. Consequently, for a given material, $p_0 \sqrt{l}$ should be constant (in complete conformity with (2.1)).

Griffiths' experiments (see Table) thoroughly confirmed that this value was a constant and thus confirmed the theoretical system advanced above.

The experiments of Roesler [53] and Benbow [54] in which stable

TABLE

Сферические колбы 1				Цилиндрические трубки 2			
$2l$ 3 (дюймы)	D 3 (дюймы)	p_0 4 (фунты/дюйм²)	$p_0\sqrt{l}$	3 $2l$ (дюймы)	3 D (дюймы)	4 p_0 (фунты/дюйм²)	$p_0\sqrt{l}$
0.15	1.49	864	237	0.25	0.59	678	240
0.27	1.53	623	228	0.32	0.71	590	232
0.54	1.60	482	251	0.38	0.74	526	229
0.89	2.00	366	244	0.28	0.61	655	245
				0.26	0.62	674	243
				0.30	0.61	616	238

1) Spherical retorts; 2) cylindrical tubes; 3) inches; 4) lbs/in^2.

conical cracks were formed and which are notable for their elegance, are of special interest for confirming the theory of brittle fracture. Figure 13 shows the system by which these experiments were performed; a photograph of conical cracks in fused quartz taken from the article by Benbow [54] is given in Fig. 14. The cracks were generated by pressing a cylindrical steel punch with a flat end into specimens of glass [53] and fused quartz [54]. In accordance with what has been presented above, the diameter s of the base of the conical crack depended only on the diameter d_0 of the

Fig. 13.

Fig. 14.

base of the punch, the pressure P on the punch, the modulus of cohesion K, and the Poissons' ratio ν. Since the corresponding problem of elasticity theory is naturally formulated so that Young's modulus is eliminated, it is not necessary to include Young's modulus among the determining parameters. Dimensional analysis gives

$$s = \left(\frac{P}{K}\right)^{2/3} \varphi\left[\frac{K^{2/3} d_0}{P^{2/3}}, \nu\right] \qquad (4.14)$$

Here, φ is some nondimensional function of its own arguments.

Experiments carried out with punches of three diameters on eleven glass samples [53] thoroughly verified the existence of the universal relationship (4.14). It follows from (4.14) that for large P, when the effect of the first argument of the function φ becomes negligibly small, "self-modeling" arises and the following equation is satisfied:

$$s = \left(\frac{P}{K}\right)^{2/3} \varphi_1(\nu) \tag{4.15}$$

Figure 15 shows a graph taken from the article by Benbow [54], showing s as a function of P according to the aforementioned experiments with fused quartz carried out under conditions corresponding to the self-modeling regime. As may be seen, the experiments being considered convincingly confirm Eq. (4.15) and thereby the system presented above.

The experiments described were carried out on materials which can be considered to be totally brittle. This is expecially true of fused quartz. Benbow [54] cites some facts indicating that the mechanism of crack formation in fused quartz is closer to pure brittle fracture than is this mechanism in glass: cracks in glass grow in size for a long time under constant load, while cracks in fused quartz rapidly increase in size during the same time and then remain constant; after the load is relieved, the cracks in the glass remain clearly visible, while those in quartz are unnoticeable, etc.

However, the significance of the theory of brittle fracture was found to go far beyond the limits of its applicability to the comparatively rare totally brittle material. Experimental studies have shown that when cracks are formed, some materials which seem to be completely plastic in ordinary tests fail in such a way that plastic deformation, while it does occur, is centered in a thin layer in the vicinity of the surface of the crack.

Thus, Fehlbeck and Orowan [28] conducted experiments on the failure of low-carbon steel plates with applied cracks under conditions corresponding to Griffiths' system of uniform elongation. The results of the experiments agreed well to Griffiths' formula but the magnitude of the surface-energy density determined from these experiments proved to be approximately three orders of magnitude greater than the surface tension of the material studied. It exhibited satisfactory correspondence with the specific work of plastic deformation in the layers of the crack near the surface as determined by independent measurements.

Basing their work on this and other analogous experimental results, Irwin [23] and Orowan [24] introduced the concept of quasibrittle fracture, making it possible to expand greatly the limits of applicability of the theory of brittle fracture. According to this concept, the theory of brittle fracture was extended to cases where plastic deformation is centered in a thin surface layer of the crack. In this case, the energy T expended in the formation of a unit of crack surface is expressed as the sum of the specific work involved in overcoming the molecular cohesive forces — the surface tension T_0 — and the specific work T_1 expended on plastic deformation.

$$T = T_0 + T_1 \tag{4.16}$$

The formal extension of the approach presented above to quasibrittle fracture was carried out in the following fashion (Fig. 16; the area of plastic deformation near the surface is crosshatched). We imagine the entire plastic region to have been cut out and the end of the crack to have been transferred to the end of the plastic region. This can be done if we assume that the forces acting from the plastic region to the elastic region are external forces applied to the surface of the crack. After this, all the

Fig. 15. Fig. 16.

preceding considerations involved in the assumption that the plastic region is thin remain unchanged and, if we again use the hypothesis of a stable terminal region of the crack surface (which also includes the boundary between the elastic and plastic regions), the modulus of cohesion is expressed in the following fashion:

$$K = \int_0^{d+d'} \frac{G(t)\,dt}{\sqrt{t}} = \sqrt{\frac{\pi E T}{1-\nu^2}} \qquad (4.17)$$

Here G(t) is the distribution of normal stresses acting at the boundary between the elastic and plastic regions.

When it is possible to disregard the contribution of the molecular cohesive forces to the integral (4.17) in comparison to the contribution of the stresses acting in the region in front of the actual end of the crack and having the order of magnitude of the yield point σ_0, we obtain an estimate of the modulus of cohesion

$$K = \sqrt{\frac{\pi E T_1}{1-\nu^2}} \approx 2\sigma_0 \sqrt{d'} \qquad (4.18)$$

Let us emphasize that the yield point σ_0 in the vicinity of the end of the crack can differ from the yield point obtained in tensile-testing large specimens.

The concept of quasibrittle fracture is similar to some extent to the concept of the "plastic particle" at the ends of a groove with a zero radius of curvature which was introduced in the classic monograph by Neuber [80].

We shall speak further of cracks formed in brittle fracture, bearing in mind the possibility of extending the results obtained to the case of quasibrittle fracture. It is understood that it is necessary in this case to consider the irreversibility of cracks formed in quasibrittle fracture as definite.

7. Cracks in thin plates. For thin plates where it is possible to assume that a plane stressed state exists, all equations derived for the case of plane deformation hold true if we replace E by $(1-\nu^2)$ and assume that the modulus of cohesion has some other value K_1. Repeating the derivation of Formula (4.13) for a plane stressed state, we obtain

$$K_1^2 = \pi ET \tag{4.19}$$

Let us note that, as experiments have shown, the surface-energy density T in the case of quasibrittle fracture increases somewhat with a reduction in the thickness of the plates [48]; this is explained by the expansion of the region of plastic deformation near the surface. The work by Frankland [8] makes an attempt at approximate theoretical calculation of this phenomenon.

Of these two cases, having in mind the complete analogy of the formal investigation of the plane stressed state and plane deformation, we shall consider only plane deformation further.

V. Specific Problems of the Theory of Equilibrium Cracks

In this section we shall consider the solutions available at the present time for various specific problems in the theory of cracks. Individual examples will be of illustrative character and the majority of the problems cited will be of interest by themselves.

1. Isolated rectilinear cracks. In this paragraph and the next, we shall study isolated mobile-equilibrium cracks over whose entire contour the maximum cohesive-force intensity is reached. For these cracks, the problem reduces to determination of the crack contours corresponding to the given load in such a way that Condition (4.6) is satisfied on these contours, and is a particular case of problem B formulated above. It is assumed that the initial cracks permit formation of such cracks; the necessary requirements imposed on the initial cracks for reversible or irreversible cracks are easily deduced from the solutions obtained.

Let us consider, under the conditions of plane deformation, an isolated rectilinear mobile-equilibrium crack in an infinite body, the crack extending along the x axis from x = a to x = b. Let p(x) be the distribution of normal stresses arising at the site of the crack in a continuous body under the same loads. This distribution is determined by the general methods of elasticity theory and we will assume it to be given. It is possible to show, using the solution advanced by N.I. Muskhelishvili [2, 18] that the tensile stresses in the vicinity of the ends of the crack, calculated without considering the cohesive forces, go to infinity according to the rule

$$\sigma_y = N / \sqrt{s} + \ldots$$

where

$$N_a = \frac{1}{\pi\sqrt{b-a}}\int_a^b p(x)\sqrt{\frac{b-x}{x-a}}\,dx, \quad N_b = \frac{1}{\pi\sqrt{b-a}}\int_a^b p(x)\sqrt{\frac{x-a}{b-x}}\,dx \qquad (5.1)$$

are the values of the coefficient of stress intensity at points a and b respectively. Satisfying Condition (4.6) at these points, we obtain expressions which determine the coordinates a and b of the ends of the crack in the form

$$\int_a^b p(x)\sqrt{\frac{b-x}{x-a}}\,dx = K\sqrt{b-a}, \qquad \int_a^b p(x)\sqrt{\frac{x-a}{b-x}}\,dx = K\sqrt{b-a} \tag{5.2}$$

In particular, if the load applied is symmetrical relative to the center of the crack, at which it is convenient to locate the coordinate origin $-$ $\underline{a}$ = $\underline{b}$ = $\underline{l}$ and Eqs. (5.2) reduce to a single expression which determines the half-length $\underline{l}$ of the crack:

$$\int_0^l \frac{p(x)\,dx}{\sqrt{l^2-x^2}} = \frac{K}{\sqrt{2l}} \tag{5.3}$$

Let us emphasize that since p(x) is an assigned function, (5.2) and (5.3) are terminal equations. These equations determine the positions of the ends of an isolated rectilinear mobile-equilibrium crack at the load in question if this load makes it possible for such a crack to exist.

Masubuchi [82] has pointed out a method for calculating the rate of elastic-energy liberation $\partial W/\partial \underline{l}$ for an isolated symmetrical crack based on a trignometric representation which he proposed for the stresses p(x) and shears $\underline{v}$ at points on the surface of the crack.

$$p(x) = \frac{E}{4l}\sum_{n=1}^{\infty} nA_n\frac{\sin n\theta}{\sin\theta}, \quad v = \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin n\theta, \quad x = l\cos\theta \tag{5.4}$$

As Masubuchi showed,

$$\frac{\partial W}{\partial l} = \frac{E\pi}{8l(1-\nu^2)}\sum_{n=1}^{\infty}(nA_n)^2 \tag{5.5}$$

Equating this expression to 4T, where T is the surface energy density, we can obtain an expression which relates the stresses applied to the size of the crack, but in a form far more complex than (5.3).

Let us analyze certain examples. Let a crack be kept open by a uniform tensile stress applied at infinity. As we have already noted, this problem was first considered by Griffiths [3, 4]. In this case, $p(x) \equiv p_0$ and Eq. (5.3) yields

$$l = \frac{2K^2}{\pi^2 p_0^2} \tag{5.6}$$

Equation (5.6) is represented by the broken line in Fig. 17. As may be seen, the size of the mobile-equilibrium crack decreases with increasing tensile stress and this indicates that the crack in this case is an unstable mobile-equilibrium crack. Despite this instability, the dimension l as determined from Eq. (5.6) has physical significance. To be more precise, if there was a crack with a length $2l_0$ in a body to which a constant tensile stress p_0 is applied at infinity, this crack will not widen when $l_0 < l$, (and closes in the case of reversible cracks), while when $l_0 > l$, it widens without limit. Thus, the dimension l plays the role of a critical dimension (for a more detailed treatment of this, see Section V, Paragraph 3).

Fig. 17.

It is obvious that in this case the instability of mobile equilibrium corresponds fully to the essence of the matter and, notwithstanding the opinion expressed by Ya.I. Frankel' [5], is not due to Griffiths' incorrect representation of the geometry of the crack ends.

If there is no stress at infinity and the crack is maintained by applying a uniformly distributed pressure to portions of its surface ($0 \leq x \leq l_0$), while the rest of the crack surface ($l_0 < x$ $x \leq l$) is free of stresses, the half-length l of the mobile-equilibrium crack is determined [58] from the expression:

$$\sqrt{\frac{l}{l_0}} \arcsin\left(\frac{l_0}{l}\right) = \frac{K}{p_0 \sqrt{2l_0}} \tag{5.7}$$

Equation (5.7) is represented in Fig. 17 by solid lines obtained from one another by a transformation of similitude. It is obvious that the opening of a crack, i.e., the appearance of an open section in it, is possible only if l_0 is no less than the corresponding size of a mobile-equilibrium crack kept open by a uniform tensile stress P_0 at infinity, as determined by Eq. (5.6). Consequently, all solid lines in Fig. 17 begin at the broken line.

The limiting case of Eq. (5.7) is curious, corresponding to p_0 going to infinity while l_0 goes to zero, so that $2p_0l_0 \equiv$ const = = P. This case corresponds to a crack kept open by concentrated forces applied to opposing points on its surface. The half-length of the crack is determined here by the expression

$$l = \frac{P^2}{2K^2} \qquad (5.8)$$

We should note that (5.6) and (5.8) can be obtained correctly to within a constant multiplier by recourse to dimensional analysis. Actually, for example, the size of a crack kept open by concentrated forces is determined only by the magnitude P of these forces and by the resultant cohesive-force characteristic, the modulus of cohesion K. It is obvious that the modulus of elasticity and Poisson's ratio are not included among the determining parameters, since the corresponding problem of elasticity theory is naturally formulated in terms of stresses so that these factors are not included among the determining parameters. Considering the dimensions of P and K, we see that only one combination with the dimensions of length can be formed from these quantities — the ratio P^2/K^2 — and that it is impossible to construct any nondimensional combinations. Thus, by virtue of the basic theorem of dimensional analysis [59], the length of a mobile-equilibrium crack should be proportional to P^2/K^2, while the coefficient of proportionality should be a universal constant in full conformity with Eq. (5.8).

Further, let the crack be maintained by two equal and opposite concentrated forces P whose points of application are at a distance L along the common line of action of the forces; it is assumed that the crack is perpendicular to the line of action of the forces and that it is symmetrical [58].

In this case, the distribution of tensile stresses at the site of the crack in the solid has the form

$$p(x) = \frac{PL}{2\pi(x^2 + L^2)}\left[1 - \nu + 2(1+\nu)\frac{L^2}{x^2+L^2}\right] \qquad (5.9)$$

(the origin of the coordinates is located at the center of the crack, at the point of intersection of the line of the crack and the line of action of the forces). Applying Eq. (5.3), we obtain an expression which determines the size of the crack in the following form:

$$\frac{P}{K\sqrt{L}} = \left(1 + \frac{L^2}{l^2}\right)^{3/2} \frac{\sqrt{2}}{[2 + (3+\nu)L^2/l^2]\sqrt{L/l}} \qquad (5.10)$$

Figure 18 is a graph of $P/K\sqrt{L}$ as a function of the relative length of the crack l/L for $\nu = 0.25$. As may be seen, when $P > P_0$, each value of P corresponds to two lengths of the mobile-equilibrium crack. Here, when P is increased, the shorter length decreases and the greater length increases. The state of mobile equilibrium corresponding to the shorter length is unstable; the corresponding branch of the curve of load as a function of length is represented by the broken line in Fig. 18. The states of mobile equilibrium corresponding to the greater length are stable; the corresponding branch is represented by a solid line. The shorter length l_1 plays the critical role for a given load P, so that initial cracks in a body whose length is less than $2l_1$ do not widen under the action of an applied force of magnitude P (close in the case of reversible cracks), while those of greater length widen until the crack reaches the second (stable) equilibrium size*. When $P < P_0$, Eq. (5.10) has no solution, i.e., no solution to the problem being considered exists. This means that whatever the length of the original crack, it does not widen under the given load to form a mobile-equilibrium crack. The critical value P_0 of the forces corresponds to the dimension l_0 of the mobile-equilibrium crack and is not zero.

The work of Romualdi and Sanders [52] dealt with the interesting problem of the effect of riveted stiffening ribs on crack propagation. This problem is schematized by the authors in the following fashion (Fig. 19). An infinite plate is tensioned by a uniform stress p_0 in a direction perpendicular to the crack. The action of

Fig. 18.

Fig. 19.

the rivets and the stiffening ribs is represented by two symmetrically distributed pairs of opposing concentrated forces with magnitudes equal to P and may be assumed to be given (in order to simplify the problem somewhat).

Substituting the corresponding stress distribution in Eq. (5.3) and computing** the elementary, although somewhat unwieldy, integrals, we obtain the relation between the applied load and the half-length l of the equilibrium crack:

$$\frac{p_0\sqrt{L}}{K} = \frac{\sqrt{2}}{\pi}\frac{P}{K\sqrt{L}}\bar{y}_0\left[\frac{1-\nu}{A\sqrt{A-B+2}} + \frac{12(1+\nu)\bar{y}_0^2}{A^2(A+B-2)\sqrt{A-B+2}} + \right.$$
$$\left. + \frac{2(1+\nu)(2B-A-4)}{A^2\sqrt{A-B+2}} + \bar{y}_0^2\,\frac{(1+\nu)(B+A)(2B-A-4)}{A^3(A+B-2)\sqrt{A-B+2}}\right] + \frac{\sqrt{2}}{\pi\sqrt{\bar{l}}} \qquad (5.11)$$
$$\bar{y}_0 = \frac{y_0}{L}, \quad \bar{l} = \frac{l}{L}, \quad B = \bar{y}_0^2 + \bar{l}^2 + 1, \quad A = \sqrt{B^2 - 4\bar{l}^2}$$

The results of the calculation are shown graphically in Fig. 20 for $\nu = 0.25$, $P/K\sqrt{L} = 0.2$, and several values of the parameter y_0/L. As may be seen, when there are no stiffening ribs, mobile-equilibrium cracks are unstable. The effect of the stiffening ribs is manifested first of all in an increase in the size of the mobile-equilibrium crack at the given load and, what is especially important, in the appearance of stable mobile-equilibrium states at rather low values of y_0/L, i.e., when the rivets are quite close together. The appearance of stable mobile equilibrium states substantially alters the character of crack development (for greater detail, see below).

The authors experimentally observed transitions of a crack from unstable mobile-equilibrium states to stable states; their experiments, which were carried out on aluminum-alloy plates both with and without stiffeners, showed a considerable increase in the sizes of mobile-equilibrium cracks at constant p_0 due to the presence of the ribs. Reference [52] also gives an experimental determination of the coefficient of stress intensity at the ends of the crack for several stable and unstable mobile-equilibrium states. In the absence of stiffening ribs, the measurements of the coefficient of stress intensity were carried out directly, on the basis of the tensile-stress decline in the vicinity of the ends of the crack (at distances known to be larger than the size of the terminal region of the crack). In the presence of stiffening ribs, the coefficients of stress intensity were measured indirectly. Satisfactory agreement of the intensity coefficients was obtained in all cases with the exception of two instances where the intensity coefficients were approximately 15% smaller. However, these two experiments, which were carried out on the same specimen – in one case on a stable and in the other on an unstable crack – gave closely similar values for the coefficients of stress intensity. (The somewhat lower value of the coefficient of intensity at the end of the stable crack can be explained by the considerable dynamic effects noted by the authors in the transition from this state to the unstable state.) Consequently, it can be assumed that the discrepancy observed was the result of some peculiarity of the particular specimen. Thus, these experiments are a direct confirmation of the general system developed above.

This investigation can be extended directly to rectilinear cracks in an anisotropic medium which lie in the planes of elastic symmetry of the material. Wilmore [21] and Stroh [83] dealt with the problem of a rectilinear crack in an orthotropic infinite body subject to a uniform stress field. Reference [83] also extended the results obtained in [16] for a rectilinear crack in an anisotropic body acted upon by an arbitrary stress field, and also found the coefficients of stress intensity at the ends of the crack for this problem. Reference [84] presented a solution of the general problem of a rectilinear mobile-equilibrium crack in an orthotropic body acted upon by an arbitrary stress field symmetrical about the line of the crack.

2. Axially symmetrical plane cracks. If a discoid mobile-equilibrium crack with a radius R is maintained in an infinite body acted upon by a certain axially symmetrical load, the tensile stresses in the vicinity of the crack outline as calculated without

considering the cohesive forces, goes to infinity according to the law

$$\sigma_y = \frac{N}{\sqrt{s}}, \quad N = \frac{1}{\pi\sqrt{R/2}}\int_0^R \frac{rp(r)\,dr}{\sqrt{R^2 - r^2}} \tag{5.12}$$

where p(r) is the distribution of tensile stresses at the site of the crack in a continuous medium acted upon by the same loads. According to the general Eq. (4.6), the equation which determines the radius R of a mobile-equilibrium crack takes the form

$$\int_0^R \frac{rp(r)\,dr}{\sqrt{R^2 - r^2}} = K\sqrt{\frac{R}{2}} \tag{5.13}$$

This equation was established in [56, 57]; its derivation was based on the use of a method of solving axially-symmetrical problems in elasticity theory using the Fourier-Hankel integral transformation, as developed in the work by Sneddon [14, 15]. In particular, if a mobile-equilibrium crack is kept open by a uniform tensile stress p_0 at infinity, $p(r) \equiv p_0$ and the radius of the mobile-equilibrium crack is determined by the expression

Fig. 20.

$$R = \frac{K^2}{8p_0^2} \tag{5.14}$$

This problem was first solved by Sack [20] by an energy method completely analagous in principle to the corresponding plane problem considered by Griffiths [3, 4].

If there is no tensile stress at infinity and the crack is kept open by a uniformly distributed pressure p_0 along the part of its surface $0 \leq r \leq r_0$, while the rest of the crack surface $r_0 < r \leq R$ is free, the radius of the mobile-equilibrium crack is determined from the relationship

$$\frac{p_0\sqrt{r_0}}{K}=\frac{1}{\sqrt{2}}\left(\frac{r_0}{R}\right)^{-3/2}\left[1+\sqrt{1-\left(\frac{r_0}{R}\right)^2}\right] \tag{5.15}$$

In this case, as in the case of a plane crack, the radius r_0 of the curved part of the crack surface should not be less than the critical radius determined from Eq. (5.14) for the pressure p_0.

In particular, if a discoid crack is kept open by equal and opposed concentrated forces P applied to its surface, the radius of the mobile-equilibrium crack is determined by the formula

$$R=\left(\frac{P}{\sqrt{2\pi}K}\right)^{2/3} \tag{5.16}$$

In complete analogy to the plane cracks, Eqs. (5.14) and (5.16) can be obtained correct to within a constant nondimensional multiplier by dimensional analysis.

If a discoid crack is kept open by equal and opposed forces P whose points of application along the common line of action of the forces differ from one another by a distance 2L, the radius R of the mobile-equilibrium crack is determined from the equation

$$\frac{P}{KL^{3/2}}=\pi\sqrt{2}\left(\frac{L}{R}\right)^{-3/2}\left(1+\frac{L^2}{R^2}\right)^2\left(1+\frac{2-\nu}{1-\nu}\cdot\frac{L^2}{R^2}\right)^{-1} \tag{5.17}$$

The solutions presented were given in [56]; the explanation of the equations obtained is completely analogous to the corresponding cases for rectilinear cracks.

3. Study of the development of isolated cracks with proportional loading. Stability of isolated cracks. Under this heading we shall consider the problem of the development of a given isolated initial crack under proportional loading — a particular case of problem A. A complete study is to be made for the case of symmetrical loads and initial cracks and simultaneously for rectilinear and discoid cracks. Let us consider an example of the problem of development of a nonsymmetrical initial crack which will

cast light upon the procedure for solving this problem.

With proportional loading, the tensile stresses at the crack site in a continuous medium under a given load are proportional to the loading parameter λ, so that $p(x) = \lambda f(x)$ and $p(r) = \lambda f(r)$ in the cases of rectilinear and discoid cracks, respectively. Introducing the nondimensional variable ξ, which equals x/l and r/R, respectively, for these cases, we may reduce Eqs. (5.3) and (5.12) to the form

$$\frac{\sqrt{2}\lambda}{K} = \varphi(c) \tag{5.18}$$

where $\varphi(c)$ is determined by the following equations in the respective cases:

$$\varphi(c) = \left[\sqrt{c}\int_0^1 \frac{f(c\xi)\,d\xi}{\sqrt{1-\xi^2}}\right]^{-1}, \quad \varphi(c) = \left[\sqrt{c}\int_0^1 \frac{f(c\xi)\,\xi\,d\xi}{\sqrt{1-\xi^2}}\right]^{-1} \tag{5.19}$$

and c denotes either the half-length l of the crack or the radius R of the crack. Thus, the dependence of crack length on the proportional-loading parameter λ is determined completely by the length of the initial crack and the function $\varphi(c)$ corresponding to the load distribution in question.

We can obtain definite results for the behavior of the function $\varphi(c)$ with the most general assumptions. We shall not consider cases where the crack is kept open by concentrated forces applied to its surface, but assume that the crack is kept open by loads of any type — e.g., by concentrated forces applied from within the body and perhaps by distributed loads applied to the surface of the crack. In this case, the functions $p(x)$, $p(r)$, and consequently, $f(c\xi)$ are definitely bounded. For small c we obtain from (5.19) the expressions

$$\varphi(c) = \frac{2}{\pi f(0)\sqrt{c}} + \ldots, \qquad \varphi(c) = \frac{1}{f(0)\sqrt{c}} + \ldots \tag{5.20}$$

respectively.

Let us assume that the tensile loads applied to the body on each side of the crack are bounded and, more precisely, equal to λP. We then have the epxressions

$$\int_{-\infty}^{\infty} p(x)\,dx = \lambda P, \qquad \int_{0}^{\infty} f(c\xi)\,d\xi = \frac{P}{2c}$$
$$\int_{0}^{\infty} p(r)\,r\,dr = \frac{\lambda P}{2\pi}, \qquad \int_{0}^{\infty} f(c\xi)\,\xi\,d\xi = \frac{P}{2\pi c^2} \qquad (5.21)$$

From these and from (5.19), we obtain asymptotic representations with $c \to \infty$ for the functions $\varphi(c)$:

$$\varphi(c) = \frac{2\sqrt{c}}{P} + \ldots, \qquad \varphi(c) = \frac{2\pi c^{3/2}}{P} + \ldots \qquad (5.22)$$

Thus, under the assumptions which we have made, $\varphi(c)$ tends to infinity when $c \to 0$ and $c \to \infty$. By virtue of the bounding of $f(c\xi)$, the integrals in Expressions (5.19) do not tend to infinity for any <u>c</u>, and thus $\varphi(c)$ has no increasing segments and, consequently, has at least one positive minimum and at least one increasing and one decreasing segment. If the forces applied to the body on each side of the crack are unbounded, the function $\varphi(c)$ has no increasing segments and, consequently, no minima. This is true, in particular, in the case of a uniform tensile stress field where $p = \lambda p_0$ and

$$\varphi(c) = \frac{2}{\pi p_0 \sqrt{c}}, \quad \varphi(c) = \frac{1}{p_0 \sqrt{c}} \qquad (5.23)$$

respectively, for rectilinear and axially symmetrical cracks.

By definition, an equilibrium crack is stable if no rather small change in its contour leads to the creation of forces which tend to remove the body still further from its disturbed equilibrium state. It is obvious that stationary-equilibrium cracks are always stable. In order for mobile-equilibrium cracks to be stable, it is necessary that their size increase with increasing loading parameter λ. Actually, we assume that the corresponding dimension of the mobile-equilibrium crack $\underline{c}$ increases when the load increases. If we increase the length of the crack while leaving the load unchanged (λ = const), the force pulling the crack apart will be larger than the equilibrium force. Consequently, equilibrium will be disturbed and the action of the excessive force on the crack will tend to widen it. On the other hand, if the size of the crack is somewhat greater than its equilibrium value, the equilibrium is displaced in the other direction and the crack, if it is reversible*, tends to close. If, on the other hand, the equilibrium dimension $\underline{c}$ of the crack decreases with increasing parameter λ when the crack is near a given equilibrium state, it is obvious that a small change in this dimension at constant load will give rise to forces which create a further deviation from the equilibrium state. The corresponding equilibrium state will be unstable. Thus, the equilibrium state of a crack is stable if it satisfies the following condition for a given $\underline{c}$ and λ:

$$\frac{dc}{d\lambda} > 0 \tag{5.24}$$

Differentiating (5.18) with respect to λ,

$$\frac{dc}{d\lambda} = \frac{\sqrt{2}}{K\varphi'(c)}. \tag{5.25}$$

From this and (5.24) we obtain the condition for stability for a mobile equilibrium state in the form

$$\varphi'(c) > 0 \tag{5.26}$$

Thus, only those states of mobile equilibrium which correspond to rising segments of the curve of $\varphi(c)$ are stable.

We now have everything necessary for complete study of the development of an isolated symmetrical crack under proportional loading. Let our system of loads applied to the body correspond to the function $\varphi(c)$ shown in the graphs in Fig. 21.

Let us first consider the case where $\varphi(c) \to \infty$ as $c \to \infty$ (Fig. 21a). In particular, this case occurs when the loads applied to both sides of the crack are bounded. Let the size of the initial crack c_1 correspond to the unstable branch of $\varphi(c)$. Then, as the parameter λ increases, the length of the crack remains constant until 2 reaches a value at which the initial crack of size c_1 becomes a mobile-equilibrium crack. Since mobile equilibrium is unstable, the crack then begins to widen under constant load until the next stable equilibrium state is reached*. With a further increase in λ, the size of the crack increases continuously until a load is reached which corresponds to the maximum of $\varphi(c)$, again changes jumpwise on transition to another stable branch, and then increases continuously with increasing λ. The path of the point representing the variation of the crack is shown in Fig. 21a and designated by the numeral 1.

Now let the dimension c_2 of the initial crack correspond to the stable branch of (c). Then the dimension of the crack remains unchanged until a load is reached at which the crack becomes a mobile-equilibrium crack, after which it increases continuously. The path of the representative point is shown in Fig. 21a and keyed there by the numeral 2.

As may be seen, the body does not fail at any value of the parameter λ in the case under consideration. If λ is less than its critical value, which corresponds to the lowest of the minima of $\varphi(c)$, it will not widen under the action of this load, however large the initial crack. The dimension corresponding to this critical value of λ for a mobile-equilibrium crack is finite.

Among other things, it follows from what has been said that if a crack is kept open by forces applied from within the body and perhaps by distributed loads applied to the surface of the crack, and if the forces applied to each side of the crack are bounded, there exist a critical value for the parameter λ, and for all values of λ greater than the critical value at least one stable and one unstable mobile equilibrium state.

Fig. 21.

Let us turn to the case where $\varphi(c) \rightarrow 0$ as $c \rightarrow \infty$ (Fig. 21b). If the size c_1 of the initial crack corresponds to a stable branch of $\varphi(c)$, the crack will not widen until a load is reached at which the initial crack becomes a mobile-equilibrium crack. The crack then increases continuously with increasing λ until a value of the parameter λ is reached which corresponds to a maximum. If this maximum is exceeded in the slightest, there is no longer a solution to the problem – the body fails. The path of the representative point is shown in Fig. 21, where it is keyed by the numeral 1. If the dimension of an initial crack c_2 corresponds to the unstable right branch of $\varphi(c)$, the initial crack does not increase in size with increasing parameter λ until a value of λ is reached at which the initial crack becomes a mobile-equilibrium crack. If this value of λ is exceeded even slightly, the body fails. If the dimension of an initial crack c_3 corresponds to the unstable left branch of the curve of $\varphi(c)$, the crack widens for $c_3 < c_0$ in the same fashion as was noted for case 2; when $c_3 > c_0$, the crack develops in a manner analogous to case 1, Fig. 21a until a maximum is reached, whereupon failure of the body occurs.

For other forms of the curve $\varphi(c)$, the investigation can be carried out easily by combining the cases considered. Knowledge of the function $\varphi(c)$ makes it possible to describe exhaustively the development of an isolated symmetrical crack in an infinite body subject to proportional loading. For reversible cracks, it is possible to trace the change in the size of the crack, even with a nonmonotonic change in load, by using the curve of $\varphi(c)$. It is curious to note that in this case, the size of the crack decreases stepwise when the load decreases, generally speaking on transition through other critical equilibrium states, rather than with an increase in load.

Very recently, L.M. Kachanov [84a] conducted a study which generalized the preceding considerations for cases in which the modulus of cohesion is taken into account as a function of time. This study is of basic importance in connection with the problems of so-called "long-term strength."

The study made in this section is based on [59].

Fig. 22.

Let us now consider for one simple case the solution to the problem of the development of a nonsymmetrical initial crack. Let there be in an infinite unloaded body a rectilinear initial crack whose ends have the coordinates $x = -a_0$ and $x = b_0$ (to be more specific, let us assume that $b_0 < a_0$), and let equal and opposing concentrated forces P be applied to opposite points on the surface of the crack (points which may be assumed to correspond to $x = 0$).

The magnitude of the force P serves as a loading parameter. According to (5.1), the coefficients of tensile-stress intensity N_0 for x = -a and x = b are equal respectively to

$$N_a = \frac{P}{\pi\sqrt{b+a}}\sqrt{\frac{b}{a}}, \quad N_b = \frac{P}{\pi\sqrt{b+a}}\sqrt{\frac{a}{b}} \tag{5.27}$$

When $P < P_1$, where

$$\frac{P_1^2}{K^2} = \frac{(b_0 + a_0)\, b_0}{a_0} \tag{5.28}$$

both coefficients N_a and N_b are less than K/π, so that the crack does not develop either on the right or on the left.

When $P = P_1$, the coefficient N_b becomes equal to the quantity K/π, the crack becomes a mobile-equilibrium crack, and its end b begins to move to the right, progressing in accordance with the magnitude of the force applied in accordance with the expression

$$\frac{P^2}{K^2} = \frac{b\,(a_0 + b)}{a_0} \tag{5.29}$$

In this case, while $P < P_2$, where

$$\frac{P_2^2}{K^2} = 2a_0 \tag{5.30}$$

the coefficients $N_a < K/\pi$ and the left end of the crack does not move.

When $P = P_2$, $b = -a_0$, so that the crack becomes a symmetrical mobile-equilibrium crack, and when $P > P_2$, its development continues according to (5.8).

The development of an initial crack with a change in force is shown in the graph in Fig. 22.

4. Cracks emerging at the surface of a body. If cracks emerge at the surface of a body, it becomes difficult to obtain effective analytical solutions. This is due to the facts that the corresponding region cannot be represented on a half-plane by rational functions and that Muskhelishvili's method does not make it possible

to obtain an effective solution in finite form; it is thus necessary to resort to numerical solutions.

Fig. 23. Fig. 24.

At present, a number of numerical solutions have been obtained for the problem of cracks emerging at the boundary of a body; in all cases for which calculations were made, the mobile-equilibrium states were unstable.

Bowie [22] dealt with the problem of a system of k symmetrically distributed cracks of identical length which have emerged at the free surface of a round cutout in an infinite body (Fig. 23). The body is under tension at inifinity by a tensile stress p_0 applied on all sides. Bowie used Muskhelishvili's method to calculate the stresses and strains. Here, in order to obtain a solution in effective form, the author used an approximate polynominal representation of an analytical function which reflected the exterior of the circle and the slits running into it onto the exterior of a unit circle. In order to determine the dimensions of the mobile-equilibrium cracks, Bowie used Griffiths' energy method directly, calculating the rate of liberation of elastic energy. Numerical calculations were made in this work for one crack and for two diametrically opposed cracks; here it proved necessary, in order to insure sufficient accuracy of calculations, to retain about thirty terms in the polynominal representation of the reflecting function. The The results of Bowie's calculations for the cases $k = 1$ and $k = 2$ are shown in Fig. 24, where the broken line represents the corresponding cracks in an infinite body. It follows from these calculations that when $L/R > 1$, the fracture stress for two cracks with a circular cavity is very close to the fracture stress for a crack of length $2(L + R)$, so that the cavity itself has virtually no influence. Further, when the length of the crack is small, it is obvious that the conditions of mobile equilibrium are defined by the fracture stresses directly on the surface of the circle. As we know, in the case of uniaxial tension the greatest fracture stress at the edge of the cavity is $3p_0$, but $2p_0$ in the case of omnidirectional tension. It follows from this that the ratio of equilibrium loads in these cases should approximate 2/3, and this is also confirmed by Bowie's calculations.

The problem of a rectilinear crack emerging at the rectilinear free boundary of a half-space (Fig. 25) was considered independently and by different methods by Wigglesworth [85] and Irwin [51].

Wigglesworth [85] studied the case of arbitrary distribution of the normal and tangential stresses along the banks of a crack. With a symmetrical stress distribution, he reduced the problem to an integral equation for the complex dislocation $w(x) = u(x) + iv(x)$ of points on the surface of the crack.

$$\int_0^l L(x, t)\, w(t)\, dt = -\frac{4(1-\nu^2)}{E}\int_0^x p(x)\, dx \qquad (5.31)$$

Here $L(x, t)$ is a certain singular integral operator and $p(x) = \sigma(x) + i\tau(x)$; $\sigma(x)$ is the distribution of the normal stresses; $\tau(x)$ is the distribution of the tangential stresses. Equation (5.31) is solved in [85] by the method of integral transformations. Detailed calculations are made for cases where the surfaces of the crack and the boundary are free from stresses and tensile stresses p_0 are applied parallel to the boundary of the half-space at infinity.

For stresses in the vicinity of the end of the crack, the author obtained the following equation in this particular case:

$$\begin{aligned} \sigma_x + \sigma_y &= 1.586 \sqrt{\frac{l}{s}}\, p_0 \sin\frac{\varphi}{2} \\ \sigma_x - \sigma_y + 2i\sigma_{xy} &= -0.793 \sqrt{\frac{l}{s}}\, p_0 \sin\varphi \exp\frac{3i\varphi}{2} \end{aligned} \qquad (5.32)$$

On the continuation of the crack ($\varphi = \pi$), we find

$$\sigma_x = \sigma_y = 0.793\, p_0 \sqrt{\frac{l}{s}} + \ldots, \quad \sigma_{xy} = 0 \qquad (5.33)$$

From this and from (4.6), the expression for the length of a mobile-equilibrium crack is written in the form

$$l = \frac{K^2}{\pi^2 (0.793)^2 p_0^2} = 1.61 \frac{K^2}{p_0^2} \qquad (5.34)$$

Irwin [51] studied only the following particular case. He represented the unknown solution as the sum of three fields. The first field corresponded to the crack ($-l \le x \le l$ $y = 0$) in an infinite body subject to constant tensile stresses p_0 at infinity; the second field corresponded to the same crack under normal stresses $Q(x)$ applied to its surface symmetrically with respect to the x and y axes; the third field corresponded to the half-space $x \ge 0$ without a crack, at whose boundary $x = 0$ the normal-stress distribution $P(y)$ was applied symmetrically with respect to the x-axis. Satisfying the boundary conditions at the free boundary and the surface of the crack, Irwin obtained a system of integral equations for $P(y)$ and $Q(x)$;

$$\int_0^l Q(x)\,\frac{2\sqrt{l^2-x^2}\,y}{\pi(y^2+x^2)\sqrt{y^2+l^2}}\left(\frac{2y^2}{y^2+x^2}+\frac{y^2}{y^2+l^2}-2\right)dx+$$
$$+p_0\left(\frac{2y}{\sqrt{y^2+l^2}}-\frac{y^3}{(y^2+l^2)^{3/2}}-1\right)=P(y) \qquad (5.35)$$
$$-4\int_0^\infty P(y)\,\frac{xy^2}{\pi(x^2+y^2)^2}\,dy=Q(x)$$

which he solved by the method of successive approximations. The first approximation gives an expression for the length l of a mobile-equilibrium crack:

$$l=\frac{2K^2}{\pi^2 1.095^2 p_0^2}=1.69\,\frac{K^2}{p_0^2} \qquad (5.36)$$

which, as may be seen, differs negligibly from the more exact relationship (5.31).

Fig. 25.

Buckner [50] dealt with the problem of a single rectilinear crack emerging at the boundary of a circular cavity in an infinite body. There was no stress applied at infinity or at the edge of the cavity, and there was no tangential stress applied to the surface of the crack, but normal stresses were applied symmetrically and varied according to an arbitrarily assigned law: p(x). Like Wigglesworth [85] (Bueckner's work was done independently), Bueckner proceeded from a singular integral equation for transverse displacement of points on the surface of the crack. He constructed a single-parameter family of particular solutions to this equation corresponding to certain special distributions $p_n(x)$. In the general case, he suggested that p(x) be represented in the form of a linear combination of $p_n(x)$:

$$p(x)=\sum_{n=0}^{n=m}\alpha_n p_n(x) \qquad (5.37)$$

The coefficients a_n were determined by the method of least squares or by collocation. The coefficient of stress intensity N_0 at the end of the crack was represented by the coefficient a_n.

If the length of the crack was much smaller than the radius of the round cavity, we obtain at the limit the particular case of a rectilinear boundary considered above. In this particular case,

Fig. 26.

it follows from Buechner's calculations that when $P \equiv p_0 = \text{const}$, the expression for the length of a mobile-equilibrium crack takes the form

$$l = \frac{2K^2}{\pi^2 1.13^2 p_0^2} = 0.159 \frac{K^2}{p_0^2} \tag{5.38}$$

which is in good agreement with (5.34) and (5.36).

In [36], Buechner also dealt with the problem of a crack emerging at the surface of an infinitely long strip of finite width with an arbitrary load symmetrical with respect to the line of the crack (Fig. 25b). He showed that it is possible to replace the integral equation obtained in this case by an equation with a degenerate nucleus. Buechner's numerical solution for the particular case where the load is formed by couples with moments M applied to both sides of the crack at infinity gives the length of the mobile-equilibrium crack as a function of load, as represented by the curve in Fig. 26.

As we have already noted, mobile-equilibrium cracks are unstable in all cases considered in this section. Thus, when the loads are increased, the initial crack does not develop until it becomes a mobile-equilibrium crack, after which the body fails. Thus, in these problems, the load at which an initial crack becomes a mobile-equilibrium crack corresponds to the failure load, and this generally speaking, does not occur.

In the work by Winne and Wundt [32], some of the solutions presented in this section were used for calculation of the failure points of rotating notched disks and notched beams subjected to bending. The experiments conducted by Winne and Wundt, which were evaluated on the basis of these calculations, showed close agreement between the values of the surface-energy density T (or the modulus of cohesion K, which reduces to the same thing) determined from the angular velocity at which failure of the rotating notched disks occurred and that determined from the loads at which the notched beams failed when bent. This confirms that the quantities T and K are characteristics of the material and are independent of the type of stressed state.

5. Cracks in the vicinity of the boundaries of the body. Crack systems. The development of cracks in bounded bodies has a number of characteristic features. Difficulties of a mathematical nature make it impossible to conduct as exhaustive a study here as we did in the case of isolated cracks. However, the qualitative features and certain quantitative characteristics of this phenomenon can be completely investigated in the simplest problems that lend themselves to analytical solutions. Let us turn our attention first of all to the problem of a rectilinear crack in a strip of finite width (Fig. 27a). The crack is assumed to be symmetrical with respect to the centerline of the strip and the direction of its propagation is normal to the free boundary. The load which keeps the crack open is assumed to be symmetrical about the line of the crack and the centerline of the strip.

In solving the problem, we use the method of successive approximations developed by D.I. Sherman [86] and S.G. Mikhlin [87].

As a first approximation, we take the solution of the problem of elasticity theory dealing with the exterior of a periodic system of slits (Fig. 27b). Again denoting the distribution of fracture stresses which would obtain at the site of the cracks in a continuous body under the same loads by p(x), we obtain an equation determining the half-length l of the mobile-equilibrium crack in the form

Fig. 27.

$$\int_{-m}^{m} p[t_0(t)]\sqrt{\frac{m+t}{m-t}}\,dt = K\sqrt{\frac{\pi m}{2L}}, \qquad t=\sin\frac{\pi l_0}{2L}, \quad m=\sin\frac{\pi l}{2L} \tag{5.39}$$

In the particular case shown in Fig. 27, where the crack is kept open by equal and opposed concentrated forces P, whose points of application are located 2s apart along their common line of action, (5.39) takes the form

$$\frac{P}{K\sqrt{L}} = \frac{\sqrt{8(\alpha^2+1)\sin(\pi l/L)}}{\sqrt{\pi}\,\mathrm{ch}\,\sigma\left[1-\nu+(1+\nu)\dfrac{\sigma(2\alpha^2+1)\,\mathrm{ch}\,\sigma}{\alpha(\alpha^2+1)\,m}\right]} \tag{5.40}$$

where $a = \mathrm{ch}\,\sigma/m$, $\sigma = \pi s/2L$. In particular, when $s = 0$ and the concentrated forces are applied to the surface of the crack, (5.40) can be represented in the following fashion:

$$\frac{P}{K\sqrt{L}} = \sqrt{\frac{2}{\pi}\sin\frac{\pi l}{L}} \tag{5.41}$$

The dimension of a mobile-equilibrium crack, expressed as a function of load for a uniform tensile stress equal to P/2L at infinity has the form

$$\frac{P}{K\sqrt{L}} = \sqrt{\frac{2}{\pi}\,\mathrm{ctg}\,\frac{\pi l}{2L}} \tag{5.42}$$

The relationship (5.40) is shown in Fig. 28 for various values of σ. As usual, the solid lines represent the stable segments and the dotted lines the unstable segments. As may be seen, when $\sigma \geq \sigma_c \approx 0.5$, the curves have no stable segments, so that when the distance between the points of application of the forces exceeds $2L/\pi \approx 0.64\ L$, mobile-equilibrium cracks are always unstable. The study of the development of an isolated crack under proportional loading carried out in Section V, Paragraph 3 is completely analogous. The graph in Fig. 28 makes it possible to characterize completely the development of any symmetrical initial crack with increasing load.

The study which has been made was based on [58, 88]. The solution of the problem of elasticity theory for $s = 0$ was obtained by Irwin [54]. The solution of the problem of a periodic system of cracks under a uniform load at infinity was given by Westergaard [13], and independently by Koiter [89].

Fig. 28.

In using the first approximation, only the tangential stresses vanish on the lines of symmetry (represented in Fig. 27 by broken lines), which correspond to the edge of the strip, while the normal stresses differ from zero. In order to obtain the second approximation, the first approximation is added to the solution of the problem for a continuous strip at whose boundaries normal stresses are assigned with distribution such as to compensate the normal stresses obtained at the boundary in the first approximation. In this case, the boundary condition at the surface of the crack is no longer satisfied. To obtain the third approximation, the second approximation is added to the solution of the problem for the exterior of a periodic system of slits whose surface has an assigned normal-stress distribution equal to the difference between the assigned distribution and that obtained from the second approximation, and so forth.

Special evaluations [88] have shown that in the problem in point, consideration of the second and subsequent approximations for stable mobile-equilibrium states reduces to corrections of the order of 2.5-3% to the equations which interest us, so that we may limit ourselves to the first approximation.

In addition to the problems presented above dealing with the periodic system of cracks and the system of radial cracks emerging at the boundary of a round cavity, several other problems of crack systems related to rectilinear cracks located along a single straight line have been treated. The mathematical methods developed by N.I. Muskhelishvili [90, 18], D.I. Sherman [91], and Westergaard [13] make it possible to reduce the problem of the development of any system of cracks of this type to computation of quadratures. We shall concern ourselves here with the simplest example of the development of a system of two collinear rectilinear cracks of the same length in an infinite body under tension at infinity by uniform stresses p (Fig. 29). This problem was considered by Wilmore [21] and in the work of Winne and Wundt [32] (the authors of [32] refer to a particular report by Irwin). Wilmore's solution has inaccuracies in the initial presentation of the solution and these affected the final formulas. According to the solution given in [32], the dimensions of the cracks remain unchanged when $p < p_1$, where

$$p_1 = \sqrt{\frac{2}{b}}\,\frac{K}{\pi}\left[\frac{4\sqrt{\alpha(1+\alpha)}}{\sqrt{1-\alpha(1+3\alpha)}}\right], \quad \alpha = \frac{a}{b} < 1 \tag{5.43}$$

When $p = p_1$, the cracks reach an unstable mobile-equilibrium state, and the inner ends of the cracks then join, forming a crack of length 2b. Further development of the crack depends on whether the expression in braces is greater than or less than unity. If it is less than unity, which occurs when $\alpha < 0.085$, the dimension obtained after the inner ends of the crack join will be less than the dimension of the mobile-equilibrium crack corresponding to the load p_1. In this case, the crack will remain unchanged until a load $p_2 = \sqrt{2K/\pi}\sqrt{b}$ is reached and the body will then fail. If the expression in braces is greater than unity, complete failure of the body takes place at once when a load p_1 is reached. Assuming $b-a = 2l$ and going to the limit in (5.43) with $b \to \infty$, we obtain (5.6), as might be expected.

Fig. 29.

Reference [88] deals with the case where two identical cracks are kept open by concentrated forces applied to their surfaces. A complete study of the general case of symmetrical loading for a system of two cracks can be carried out in a manner completely analogous to the foregoing by using the expressions for the coefficients of stress intensity at the ends of the cracks $x = a$ and $x = b$.

$$N_a = \frac{\sqrt{2}}{\sqrt{a(b^2-a^2)}}\int_a^b p(t)\,t\sqrt{\frac{b^2-t^2}{t^2-a^2}}\,dt$$
$$N_b = \frac{\sqrt{2}}{\sqrt{b(b^2-a^2)}}\int_a^b p(t)\,t\sqrt{\frac{t^2-a^2}{b^2-t^2}}\,dt \qquad (5.44)$$

As may be seen from the examples which we have considered, collinear cracks are "weakened" by one another and reduce one another's stability. Ya.B. Zel'dovich drew attention to the fact that in the case of a "checkerboard" distribution of cracks (Fig. 30), the converse effect occurred. As calculations have shown, mobile-equilibrium cracks can be stable even in the case of uniform normal loads on the surface of the cracks when they have a definite position relative to one another.

Fig. 30.

Fig. 31.

We shall dwell briefly on the so-called "scale effect" in the brittle fracture of bounded bodies. Let us consider geometrically similar bodies (it is assumed that the macroscopic cracks present in these bodies are also geometrically similar) differing only in their characteristic dimensions d and their characteristic scales of applied fracture loads S. The quantity $S = S_0$, which corresponds to failure of the body, depends, assuming that fracture is brittle, only on the characteristic dimension d of the body and the modulus of cohesion K. A dimensional characterization of S can be constructed uniquely from the quantitites K and d, and it is impossible to construct any nondimensional combination. Simple relationships therefore hold for the magnitude of the fracture load:

$$S_0 = \varepsilon_1 K d^{3/2}, \qquad S_0 = \varepsilon_2 K d^{1/2}, \qquad S_0 = \varepsilon_3 K d^{-1/2} \qquad (5.45)$$

for the respective cases where S has the dimensions of force, force distributed along a line (as, for example, concentrated forces in plane deformation), and the dimensions of stress. The quantities ε are constant for a given geometrical configuration of the body. At the present time, there is a great deal of experimental data on the failure of geometrically similar bodies which make it possible to ascertain the limits of applicability of the theory of brittle fracture. In this connection, similar information can be found in the article by Wundt [92], and some new results are given in the work by Yusuff [93].

6. Cracks in rocks. For theoretical geology, considerable interest is presented by research on the development of cracks in rock massifs. Cracks can be formed in these masses by various factors of tectonic character, as well as in consequence of a number of artificial disturbances (mining, hydraulic splitting of strata, etc.).

A number of problems in the theory of cracks have been considered in conjunction with the theory of the hydraulic splitting of petroleum-bearing strata. The vertical-crack problem consists in the following. A crack in an infinite space compressed at infinity by a hydrostatic pressure is kept open by a viscous fluid pumped into the crack (Fig. 31). The basic feature of the problem is the fact that the fluid does not fill the crack completely: there is always a free section of the crack ahead of the wetted region. The fluid pressure p_0 in the wetted region of the crack can be assumed constant in first approximation, since a sharp tapering of the crack occurs at the end of the wetted region and nearly the entire fluid-pressure gradient is lost at the end of the wetted region. The problem derives its name from the fact that the crack described in the problem under consideration is located in a vertical plane, while q is the lateral pressure of the rock. In comparison with the effects of the lateral rock pressure and the fluid pressure, those of the cohesive forces can be neglected, as has been shown by the evaluations which have been made.* Equation (5.3), which determines the dimensions of the crack, takes the form

$$\int_0^l \frac{p(x)\,dx}{\sqrt{l^2 - x^2}} = 0, \qquad p(x) = \begin{cases} p_0 - q & (0 \leqslant x \leqslant l_0) \\ -q & (l_0 < x \leqslant l) \end{cases} \tag{5.46}$$

From this we obtain

$$l = l_0 \left[\sin \frac{\pi q}{2 p_0}\right]^{-1} \tag{5.47}$$

The expression for the maximum half-width v_0 of the crack takes the form

$$v_0 = \frac{8(1 - \nu^2) p_0 l_0}{\pi E} \ln \operatorname{ctg} \frac{\pi q}{4 p_0} \tag{5.48}$$

As the calculations show, with the values of l_0/l near unity that are usually encountered the aperture of the crack remains virtually constant over the entire extent of the wetted region of of the crack; in the free section, the crack narrows rapidly. The vertical-crack problem was first formulated and solved in the work by Yu.P. Zheltov and S.A. Khristianovich [38].

The horizontal-crack problem [40] takes the following form. A horizontal discoid crack is formed in a heavy half-space at a certain depth H, again by pumping a viscous fluid; the surface of the crack is again divided into wetted ($0 \leq r \leq R_0$) and free ($R_0 < r \leq R$) parts; the fluid pressure p in the wetted part can be assumed constant. As in the preceding case, the cohesive forces are neglected. On the assumption that the depth of the crack H is sufficiently large, the boundary condition at the boundary of the half-space is disregarded. The finite-stress condition at the boundary of the crack gives in this case

$$\frac{p-\gamma H}{p} = \sqrt{1-\left(\frac{R_0}{R}\right)^2} \qquad (5.49)$$

where γ is the specific gravity of the rock. For the volume of fluid pumped, we obtain the expression

$$V = \frac{4(1-\nu^2)pR^3}{E}\varphi\left(\frac{R_0}{R}\right), \quad \varphi(z) = z^3\left[\frac{2}{3} - \frac{z}{3} - \frac{z}{3(1+\sqrt{1-z^2})}\right] \qquad (5.50)$$

Actually, $z = R_0/R$ approximates unity, so that we can use the asymptotic form of (5.50)

$$V = \frac{4(1-\nu^2)pR^3}{3E}\sqrt{2(1-z)}\{1+\sqrt{2(1-z)}-3(1-z)\} \qquad (5.51)$$

The maximum half-width of the crack is determined by the formula

$$r_0 = \frac{8(1-\nu^2)pR_0}{\pi E}\arccos\left(\frac{R_0}{R}\right) \qquad (5.52)$$

Thus, knowing the depth at which the crack occurs, the fluid pressure, and the specific gravity of the rock being split, we can find R_0/R from (5.49); from this and (5.51), knowing the total volume of pumped fluid V, we can obtain the radius R of the crack;

then the determination of the remaining crack parameters presents no difficulty.

Reference [40, 41] also dealt with problems of horizontal cracks in a radially-variable pressure field created by the overlying rock. In this case, complete wetting of the crack surface can occur under certain conditions, i.e., this surface may have no free segment.

Yu.P. Zheltov [43] indicated an approximate method of solving the horizontal-crack problem in a vertical-pressure field which varies along the radius. Comparison of the results of calculations obtained by this method with exact solutions for certain cases has shown completely satisfactory agreement.

Using the method of successive approximations, Yu. A. Ustinov [94] evaluated the influence of the free boundary in the horizontal-crack problem. It was found that if the depth is greater than twice the radius of the crack, the influence of the free boundary is negligibly small.

The problem of a crack formed by wedging a heavy space with a horizontal wedge of constant thickness is considered in [39] in conjunction with the problem of roof settling in mining stopes.

Yu.P. Zheltov [42] generalized the solution of the vertical-crack problem to the case where the rock being split is permeable and the pumping fluid is filtering through the rock.

VI. Wedging. Dynamic Problems of the Theory of Cracks.

1. Wedging an infinite body. By wedging, we mean the formation of a crack in a solid body by the insertion of a rigid wedge into it. The most characteristic property of the wedging of a brittle body is the fact that the surface of the wedge is never in complete contact with the body: there is always a free segment at the front

of the wedge and an open crack is formed ahead of it, meeting at at some distance from the foremost point of the wedge (Fig. 32).

The problem of wedging an infinite body with a stationary wedge [39, 58, 95] is, to all appearances, the simplest formulation of this type of problem; it can be effectively and exactly solved by the methods of elasticity theory and permits us to draw qualitative inferences as to wedging under more complex conditions.

Fig. 32.

Thus, let a uniform isotropic brittle body be wedged by a thin, symmetric, absolutely hard semifinite wedge having a thickness 2h at infinity (Fig. 32). An open crack is formed in front of the wedge and unites smoothly at a certain point O; the position of point O relative to the foremost point of the wedge C is unknown beforehand and must be determined during solution of the problem. If the wedge has a rounded point (Fig. 32a), the position of the points B and B' at which the wedge makes contact with the surface of the crack are unknown beforehand and must also be determined during solution of the problem. If, on the other hand, the wedge has a truncated point (Fig. 32b) as, for example, in the case of a wedge of uniform thickness, the position of the points at which the wedge makes contact with the surface of the crack are fully determined and correspond to the corners at the front of the wedge. However, it is obvious that the stresses at the points of contact are infinite in this case. Let us assume at first that the frictional forces at the contact surface between the wedge and the body being wedged equals zero.

On the exterior of the crack, the field of elastic stresses and strains satisfies the ordinary equations of static elasticity theory. Because of the assumption made earlier that the wedge is thin, we can carry the boundary conditions along the entire surface of the crack to the slit ZBOB'A'. Without considering the cohesive forces, the boundary conditions are represented in the form

$$\sigma_{xy} = 0, \ \sigma_y = 0 \qquad (0 < x < l_2, \ y = 0) \tag{6.1}$$
$$v = \pm f(x - l_1), \ \sigma_{xy} = 0 \qquad (l_2 < x < \infty, \ y = 0)$$

Here σ_y and σ_{xy} are the components of the stress tensor; l_1 and l_2 are the distances from the point at which the crack joins to the tip of the wedge and to the points of contact between the surface of the crack and the wedge, respectively; f(t) is a function which determines the equation of the surface of the wedge in a system of coordinates which has its origin at the tip of the wedge, i.e., the function which determines the shape of the wedge; the plus and minus signs correspond to the upper and lower faces of the slit.

As may be seen, the problem of wedging is a kind of combination of the contact problem of elasticity theory [18, 72, 73] and the problem of crack theory.

The position of the points of contact between the surface of the crack and the wedge when the wedge has a rounded front edge and the position of the points at which the crack closes relative to the tip of the wedge are determined from the following two conditions.

1st. The stresses at the points of contact between the surface of the crack and the wedge must be finite. For the contact problem, an analogous condition was first proposed in the form of a hypothesis by N.I. Muskhilishvili [96, 18] and A.V. Bitsadze [97]; it is proven in [61].

2nd. The stresses at the end of the crack are finite, or, what is the same thing, smooth union of the opposing faces of the crack occurs at its end. Since the intensity of cohesive forces at the end of the crack is at maximal the stresses in the vicinity of the end of the crack, calculated without consideration of the cohesive forces, should go to infinity according to (4.8).

The problem of wedging is a mixed problem of elasticity theory. To solve it, it is convenient to consider the singular integral equation for the compressive forces at the sides of the wedge being driven in: $\sigma_y = -\Phi(x)$. When $\Phi(x)$ is known, determination of the field of elastic elements obviously leads to the solution of the first boundary problem of elasticity theory for the region surrounding a semifinite rectilinear slit, and this can be carried out by Muskhelishvili's method ([18], §95). This solution yields an expression for the transverse displacements at the point of contact between the wedge and the surface of the crack:

$$v = \frac{4(1-\nu^2)}{\pi E} \int_{l_2}^{\infty} \Phi(\sigma^2)\, \sigma \ln \left| \frac{\sigma+\zeta}{\sigma-\zeta} \right| d\sigma \tag{6.2}$$

where $\zeta = \sqrt{x}$, where the root takes both positive and negative values, giving the shears of the upper and lower faces of the notch, respectively. The second condition in (6.1) yields the basic integral equation for this problem:

$$\int_{l_2}^{\infty} \Phi(\sigma^2)\, \sigma \ln \left| \frac{\sigma+\zeta}{\sigma-\zeta} \right| d\sigma = \pm \frac{\pi E}{4(1-\nu^2)} f(\zeta^2 - l_1) \tag{6.3}$$

which, as may be shown, is equivalent to the singular integral equation obtained from (6.3) by differentiating it with respect to ζ:

$$\int_{|\sigma|>l_2} \frac{\Phi(\sigma^2)\, \sigma\, d\sigma}{\sigma-\zeta} = \pm \frac{\pi E}{2(1-\nu^2)} \zeta f'(\zeta^2 - l_1) \tag{6.4}$$

and to the condition

$$\Phi(x) = \frac{Eh}{2\pi(1-\nu^2)x} + O(x^{-2}) \qquad \text{for } x \to \infty. \tag{6.5}$$

where $h = f(\infty)$. Using the methods of solving singular integral equations developed in the monograph by N.I. Muskhelishvili [19], the solution to equation (6.4) can be obtained in the form

$$\Phi(x) = \frac{1}{\pi\sqrt{x(x-l_2)}}\left[A - \frac{E}{2(1-\nu^2)}\int_{l_2}^{\infty}\frac{f'(t-l_1)\sqrt{t(t-l_2)}}{t-x}\,dt\right] \tag{6.6}$$

Here A is an as yet indeterminate constant. The integral in (6.6) is known to exist because of the finite $f(\infty) = h$ and goes to zero when $x \to \infty$, and from this and (6.5) we obtain a value for the constant A

$$A = \frac{Eh}{2(1-\nu^2)} \tag{6.7}$$

For finite stresses at the points of departure $x = l_2$ in the case of a wedge with a rounded end, it is necessary and sufficient that the expression in brackets in (6.6) go to zero when $x = l_2$. This yields the first of the equations for determining l_1 and l_2:

$$h = \int_{l_2}^{\infty} f'(t-l_1)\sqrt{\frac{t}{t-l_2}}\,dt \tag{6.8}$$

Further, from this solution we obtain the following expression for the tensile stresses along the extension of the slit:

$$\sigma_y = \frac{E}{2\pi(1-\nu^2)\sqrt{(l_2-x)(-x)}}\left[h - \int_{l_2}^{\infty}\frac{f'(t-l_1)\sqrt{t(t-l_2)}\,dt}{t-x}\right] \tag{6.9}$$

From this and from (4.6) we obtain

$$h - \int_{l_2}^{\infty} f'(t-l_1)\sqrt{\frac{t-l_2}{t}}\,dt = \frac{2K\sqrt{l_2}(1-\nu^2)}{E} \tag{6.10}$$

The relationships (6.8) and (6.10) are terminal equations which determine the unknown constants l_1 and l_2 which enter into the solution.

In the particular case where the thickness of the wedge is uniform, $f(t) \equiv h$, Condition (6.8), which no longer holds true, is replaced by the relationship $l_1 = l_2$, while (6.10) gives the following expression for the length of the open crack in front of the wedge which is being driven in:

$$l_1 = l_2 = \frac{E^2 h^2}{4(1-\nu^2)^2 K^2} \tag{6.11}$$

Other particular wedge shapes are also considered in [95]: a wedge with a small curvature at its tip and a wedge with a power-law curvature. Study of the first of these examples has shown that a small curvature has a small effect on the length of the open crack in front of the wedge. Reference [95] also studied the case where dry-friction forces act on the faces of the wedge.

Fig. 33.

Fig. 34.

Reference [84] studied wedging of an anisotropic body by a semiinfinite hard wedge.

I.A. Marcuson [98] considered the problem of wedging of an infinite body by a wedge of finite length 2b (Fig. 33). In the case of constant wedge thickness 2h, the crack length $2l$ as a function of the wedge length 2b takes the form, all other conditions the same, shown in Fig. 34 (l_0 is the length of the open crack for an infinite wedge, as given by (6.11)).

Reference [95] also investigated the effect of a uniform compressive or tensile stress applied at infinity on the length of the open crack formed by wedging a body with a wedge of finite length.

The relationship (6.11) can be used for experimental determination of the coefficient of cohesion K. In order to accomplish this, a wedge made of a material substantially harder than the material being tested is driven into the latter and the length L of the open crack thus formed is measured. The modulus of cohesion can then be determined by the formula

$$K = \frac{Eh}{2(1-\nu^2)\sqrt{L}} \tag{6.12}$$

The wedge should be sufficiently long so that the boundary of the plate will have no influence; actually, the wedge should be driven in until the distance between the end of the wedge and the

Fig. 35.

Fig. 36.

end of the crack is not changed by further motion of the wedge. The plate should be wide and thick enough so that its stressed state can be assumed to correspond to plane deformation. In addition, in order to ensure that the crack is rectilinear, it is necessary to compress the specimen in the direction of crack propagation, as is recommended in the work by Benbow and Roesler [9]. (It can be shown that in this case, (6.11) and (6.12) remain unchanged.)

2. Wedging a strip. In its rigorous formulation, the solution of the problem of wedging bounded bodie[illegible] very difficult. There are only a few approximate solutions based on the use of the approximations of the simple theory of beams.

The first such solution was obtained by I. V. Obreimov [8]; this work was the very first study in which wedging was considered. In conjunction with the experiments which he conducted on the cleavage of mica, I.V. Obreimov dealt with the case where the strip being removed is thin and in contact with the wedging body at only one point (Fig. 35). In order to establish a connection between the surface tension of the mica and the shape parameters of the crack, I.V. Obreimov applied the methods of the strength of materials to this problem, regarding the chip as a thin beam. The theoretical part of the work by I.V. Obreimov was not free of errors; V.D. Kuznetsov [99] subsequently refined the calculations of this work in his book, as did M.S. Metsik [10] and N.N. Davidenkov [12] in their reports. M.S. Metsik also brought more precision to the experimental method of [8]. The use of the approximations of the theory of thin beams was justified in certain cases for determining the length of a crack. However, these approximations could not be admitted when describing the shape of a crack surface in the immediate vicinity of its end, even when the distribution of cohesive forces in the terminal region was explicitly included in the examination, as was done by Ya.I. Frankel [5]. This was due to the fact that the length of the terminal region cannot be considered great in comparison with the thickness of the chip, so that the chip cannot be regarded as a thin beam in the region where cohesive forces act.

To illustrate the approximate approach based on the methods of the simple theory of beams, let us dwell in more detail on the work of Benbow and Roesler [9]. We should note that this work explains most clearly the possibilities and limits of applicability of this approach.

The work deals with a problem formulated in the following form (Fig. 36). A strip of finite width $\underline{b}$ is wedged symmetrically, so that the crack opens up along the centerline of the strip. A compressive force Q/2 is applied to the cut end of the strip in order to ensure rectilinear crack propogation; the wedging force P creates a crack of length $\underline{l}$ and initial width $\underline{h}$.

Obtaining an expression for the elastic energy from dimensional considerations, the authors write the equilibrium condition of the crack in the form

$$\frac{T}{E} = \frac{h^2}{l}\,\Phi\left(\frac{b}{l}\right) \tag{6.13}$$

so that, for a given material, the magnitude of $h^2/\underline{l}$ should be uniquely determined by the quantity $b/\underline{l}$. The experiments described in [9], which were carried out on specimens of two different plastics, conclusively demonstrated the existence of such a single-valued relationship.

For small values of $b/\underline{l}$, i.e., for long cracks, it is possible to obtain an asymptotic form of the relationship (6.13) by considering both halves of the strip being wedged as thin beams embedded in a section corresponding to the end of the crack. In this case, the expression for the elastic energy of the strip takes the form

$$U(h, l) = \frac{3h^2B}{l^3} \tag{6.14}$$

Here B = EI, the rigidity of the beam, I = $nb^3/96$, and $\underline{n}$ is the transverse thickness of the beam. The surface energy of the crack is obviously equal to $2Tn\underline{l}$. In a mobile-equilibrium state, the variations in surface energy corresponding to small variations $\delta\underline{l}$ in the length of the crack equal the corresponding variations in the elastic energy of the strip, from which we obtain

$$-\frac{\partial U}{\partial l} = 2n \quad \text{или} \quad \frac{T}{E} = \frac{3h^2b^3}{64\, l^4} \tag{6.15}$$

Comparing the second formula in (6.15) with (6.13), we can find the asymptotic expression for $\Phi(b/l)$ for $b/l \to 0$

$$\Phi = \frac{3}{64}\left(\frac{b}{l}\right)^3 \tag{6.16}$$

From (6.15) we obtain the expression for the length of the crack in the form

$$l = \left(\frac{3\, h^2b^3E}{64\, T}\right)^{1/4} = \left(\frac{3\, h^2b^3E^2\pi}{64\, K^2(1-\nu^2)}\right)^{1/4} \tag{6.17}$$

As may be seen the length of the crack in this case is proportional only to $\sqrt{h}$, whereas the length of the crack (cf. (6.11)) is proportional to h^2 in wedging of an infinite body with a semi-infinite wedge.

The relationship (6.15) was used by Benbow and Roesler to determine the surface energy density in the plastics studied. We should note the great care exercised in the experimental study carried out in this work and the scrupulous appraisal of sources of possible error and their magnitudes.

In a recent survey by Gilman [11] we can find a detailed summary and bibliography of experimental studies of wedging.

3. Dynamic problems of the theory of cracks. Recently, problems of the dynamics of cracks have attracted considerable attention. A detailed treatment of these problems falls outside the scope of this survey and we shall thus limit ourselves here to a brief summary of the basic results achieved in theoretical research on the dynamics of cracks.

The work by Mott [36] deals with the widening process of an isolated rectilinear crack in an infinite body under the action of a uniform field* of tensile stresses p_0. On the basis of dimensional analysis, Mott obtained an expression for the kinetic energy of of the body:

$$\varepsilon = k\rho l^2 V^2 p_0^2 / E^2 \tag{6.18}$$

Here ρ is the density of the body, l is the half-length of the crack, V is the rate of expansion of the crack, and k is a dimensionless multiplier which Mott considered to be constant and left indeterminate. Complementing the static energy equation (2.1) with the derivative of the kinetic energy thus determined with respect to the length l of the crack, and assuming that the remaining terms in (2.1) retain the same form as in the static problem of Griffiths, Mott found the rate of crack expansion:

$$V = \left[\frac{\pi(1-\nu^2)}{k}\right]^{1/2}\left(\frac{E}{\rho}\right)^{1/2}\left(1-\frac{l_*}{l}\right) \tag{6.19}$$

where l_* is the critical half-length of the crack as determined by (5.6). Thus, as the crack propagates, the rate of its expansion increases, tending to a limit

$$V_0 = \left[\frac{\pi(1-\nu^2)}{k}\right]^{1/2}\left(\frac{E}{\rho}\right)^{1/2} \tag{6.20}$$

so that the limiting velocity, according to Mott, represents a definite fraction of the velocity of longitudinal-wave propagation.

In this discussion, the use of the static expression for the decrease in the elastic energy W remains ungrounded. In addition, the quantity k in (6.18) and (6.19), generally speaking, need not be constant: it may depend on l/l_*, V/c_1, and other nondimensional combinations.

In an exact formulation of the dynamic theory of elasticity, Yoffe [100] studied the problem of a rectilinear crack of <u>constant length</u> moving with a constant velocity in an infinite body under tension at infinity by a uniform stress. Despite some artificiality in the formulation of the problem, an important result of quite general significance was obtained from this work. Precisely, it was found that if the velocity of crack propagation becomes greater than a certain critical velocity, the direction of crack propagation ceases to be the direction of maximum fracture stress and the crack begins to deflect. The magnitude of the critical velocity V_1 is approximately 0.4 c_1, where c_1 is the velocity of longitudinal wave propagation in the given material (the ratio V_1/c_1 is virtually independent of the Poission's ratio ν of the material).

Roberts and Wells [101] made an attempt to evaluate the constant k which remained indeterminate in Mott's work. Using the value found for this constant, they obtained a limiting crack-propagation velocity which approximated that found by Yoffe. However, their evaluation, which was based on a solution to the static problem of elasticity theory, is too coarse. Since the rectilinear propagation of a crack was assumed to be definitely ensured in [101], the close agreement of the critical velocity found by Yoffe [100] with the limiting velocity found in [101], must be regarded as accidental.

If the rectilinearity of crack propagation is ensured in some manner (for example, by powerful compression of the body in the direction of crack propagation or by anisotropy of the material), the maximum velocity of crack propagation corresponds to the velocity of propagation of Rayleigh surface waves in the material under consideration and is approximately 0.6 c_1.

The first to affirm that the limiting velocity of crack propagation corresponds to the Rayleigh value was Stroh [102]. The heuristic proof given in this work reduces to the following. Noting correctly that the limiting velocity of crack propagation does not depend on the surface energy of the body, Stroh assumed a zero surface energy. Proceeding from this on the basis of energy considerations, Stroh came to the conclusion that the tensile stresses near the end of the crack were zero on its extension, so that crack propagation is a disturbance moving along a surface free of stress and capable of propagating only at the Rayleigh velocity. Actually, it is possible to conclude from Stroh's discussion only that the tensile stress at the very contour of the crack equals zero. However, the equality of crack-propagation velocity to the Rayleigh velocity does not follow from this fact, as shown by the following simple example. Let us take a body compressed at infinity by a hydrostatic compressive stress and wedged by a semiinfinite wedge (Fig. 32) moving at an infinitesimally small velocity. The cohesive forces and consequently, the surface energy are assumed to be zero. Because of the infinitesimally small velocity of the wedge, the dynamic effects are nonessential, so that, in accordance with Section III, Paragraph 2, we may assert that the tensile stress at the end of the crack equals zero. At the same time, the crack-propagation velocity equals the velocity of the wedge, i.e., is also infinitesimally small.

By a chain of reasoning based on analysis of exact solutions to the equations of the dynamic theory of elasticity, the conclusion as to the equality of the limiting velocity of crack propagation to the Rayleigh velocity was obtained independently and simultaneously by several authors. Craggs [103] dealt with the steady-state propagation of a semiinfinite rectilinear crack to the part to whose surface adjoining the edge symmetrically distributed normal and tangential stresses were applied. The report by An Dang Ding [104] was concerned with a nonstationary field of stresses and strains in an infinite body with a semiinfinite crack along whose surface normal symmetrical concentrated forces begin to move at constant velocity, from the edge inward, at the initial point in time. Reference [95] deals with wedging of an infinite, isotropic brittle body by a semi-infinite hard wedge of arbitrary shape moving with constant velocity. In [84], an analogous problem is studied for an anisotropic body. Baker [105] dealt with a nonstationary distribution of stresses and strains in a solid containing a semiinfinite crack to whose surface a constant normal stress is applied at the initial moment, whereupon the crack begins to propagate with constant velocity.

In the entire diversity of problems considered in these studies, the following general result, which served as the basis for the formulation of the conclusions given above, was obtained: as the characteristic velocity inherent to the problem approaches the Rayleigh velocity, a peculiar resonance phenomenon intervenes. Let us note that the appearance of reasonance on the approach to the Rayleigh velocity is not specific to the problem of cracks: investigation of the problem of a punch moving along the boundary of a half-space, which was considered by L.A. Galin [72] and Radok [106], has disclosed [95] that the same resonance phenomena arise as the punch velocity approaches the Rayleigh velocity. Apparently, the limiting character of the Rayleigh velocity is most directly demonstrated in the problem of wedging. It is obvious that the maximum possible crack propagation velocity can be achieved when the body is wedged by a moving wedge. Analysis of this problem has shown [95] that, with increasing wedge velocity, the length of the open crack in front of the wedge decreases, going to zero on the approach to the Rayleigh velocity. Thus, when the wedge moves with a velocity exceeding the Rayleigh velocity, no open crack is formed in front of it; from this it follows that the maximum velocity with which a crack can propagate equals the Rayleigh velocity.

Broberg [107, 108] dealt with the problem of a uniformly expanding crack of finite length in an infinite body subject to a uniform tensile-stress field. The solution obtained by Broberg is an asymptotic representation of the solution to the problem dealt with by Mott [36] and Roberts and Wells [101] for large time values. However, in contrast to References [36, 101], Broberg's solution was obtained on the basis of the exact methods of dynamic elasticity theory. Independently of [102-104, 57, 95, 105] and in complete agreement with the results presented in these works, Broberg found that the velocity of crack expansion in his problem equalled the limiting velocity of crack propagation for the problem considered in [36, 101] and corresponded to the Rayleigh velocity.

Let us note the works by Bilby and Bullough [109], McClintock and Suknatme [110], which dealt with uniformly moving cracks of finite and infinite length to whose surface symmetrical tangential stresses parallel to the edge of the crack were applied. In this problem, so-called "antiplane deformation" occurred instead of plane deformation when only one dislocation component, that parallel to the edge of the crack, differed from zero. Investigations of such cracks lead to solution of one wave equation (the LaPlace equation for equilibrium cracks). Cracks formed under conditions of antiplane deformation are of considerable interest as the simplest model for which an effective solution is possible for many problems insoluble for cracks formed under conditions of plane deformation because of the great mathematical difficulties involved.

An analysis of the dynamics of crack propagation based on the approximation of the simple theory of beams was made by Gilman [11] and Suits [111].

Received 28 April 1961.

[Footnotes]

Manuscript Page No.

8 *Sh.A. Sergaziyev made a very good comparison between cracks which satisfy this assumption and the popular "molniya" [lightning; zipper] fasteners.

10 *A completely analogous situation arises when the body moves along a rough horizontal surface under the action of a horizontal force. The body begins to move only after the force exceeds the maximum force of friction possible for the body and surface in question.

20 *These general formulations of the problem are useful, despite the fact that their general solution in effective form is far beyond the capabilities of contemporary mathematics. The existence of general presentations of the problem helps to clarify the importance of specific concrete solutions and of the difficulties arising in the development of the theory.

24 *In addition to these basic defects, [5] contains the errors in calculation noted in [37].

In its most general form, this convenient method of reducing the load to a load distributed over the fracture surface was originated by Bueckner [33].

35 *In the work by M.Ya. Leonov and V.V. Panasyuk [69, 70], the function f(y) was approximated by a broken line and this approximation was used as a basis for formulation of a linear integral equation for the normal displacements of points on the surface of the crack. This integral equation was then solved approximately, with a rather unfor-

tunate choice of the approximate representation of the solution, so that the shape of the crack at its end proved to be wedge-shaped with a finite end angle. Actually, as was shown above, the terminal angle is necessarily zero. Another shortcoming of these studies was the application of results obtained by methods of the mechanics of continuous media to cracks whose length was of the order of several interatomic distances.

For example, points on the contours of nonwidening notches or points on the contours of cracks produced on a decrease in load from cracks that existed under large loads.

57 *Let us note that, actually, because of the dynamic effects involved in the expansion of the initial notch, the crack can "jump" somewhat on passage through the stable equilibrium state. For more detail on this, see below.

57 **The integrals were computed and the numerical calculations made for the curve in Fig. 20 by V.Z. Parton and Ye.A. Morozova.

64 *If the crack is irreversible, the increase in its size produces no reverse closure, but further crack growth does not occur either. In this case, equilibrium is reached through a decrease in the cohesive forces acting in the terminal region of the crack.

65 *Because of the dynamic effects which arise in this transition, the crack may expand until it reaches a size which somewhat exceeds the size of the stable mobile-equilibrium crack corresponding to the load in question (apparently, just this phenomenon occurred in the experiments described in [52]). In this case, a further increase in load pro-

duces no change in the length of the crack until the instant when it becomes a mobile-equilibrium crack, whereupon it continues to expand. It is natural that the purely static theory under consideration cannot describe these dynamic effects; the corresponding segments of the graph in Fig. 21a are represented by the broken line and keyed 1'.

80 *The condition for negligibly small cohesive forces will be $K/q \sqrt{\underline{1}}$ in laboratory simulation. 1. Generally speaking, it will not be satisfied in laboratory simulation.

91 *In contrast to [36], we shall consider here plane deformation and not the plane stressed state.

BIBLIOGRAPHY

1. Inglis C. E. Stresses in a plate due to the presence of cracks and sharp corners' Trans. Inst. Nav. Arch., 1913, 55, 219—230.
2. Muskhelishvili N. I. Sur l'integration de l'équation biharmonique. Изв. Российск. Акад. наук, 1919, 13,6 сер., 663—686.
3. Griffith A. A. The phenomenon of rupture and flow in solids. Phil. Trans. Roy. Soc., 1920, A 221, 163—198.
4. Griffith A. A. The theory of rupture. Proc. 1-st. Intern. Congr. Appl. Mech..

5. Frenkel', Ya.I., Teoriya obratimykh i neobratimykh treshchin v tverdykh telakh [The Theory of Reversible and Irreversible Cracks in Solids], Zhurn. tekh. fiz. [Journal of Technical Physics], 1957, 1857-1866.

6. Smekla, A., Technische Festigkeit und molekulare Festigkeit [Technical Strength and Molecular Strength], Naturwiss [Natural Science], 1922, 10, 799-804.

7. Wolf K. Zur Bruchtheorie von A. Griffith. Zeitschr. Ang. Math. Mech.,1923, 3, 107—112.
8. Obreimov I. V. The splitting strength of mica. Proc. Roy. Soc., 1930, A 127, 290—297.
9. Benbow J. J. and Roesler F. C. Experiments on controlled fractures. Proc. Phys. Soc., 1957, B70, 201—211.

10. Metsik, M.S., Opticheskiy metod izmereniya raboty rasshchepleniya kristallov slyudy [The Visual Method of Measuring the Work of Cleavage of Mica Crystals], Izv. vuzov. ser. fiz., [Bulletin of the Higher Educational Institutions, Physics Series], 1958, No. 2, 58 - 63.

12. Davidenkov, N.N., O poverkhnostnoy energii slyudy [The Surface Energy of Mica], Prikladna mekhanika [Applied Mechanics], AN SSSR [Academy of Sciences USSR], 1960, 6, No. 2, 138-142.

13. W e s t e r g a a r d H. M. Bearing pressures and cracks. J. Appl. Mech. 1939, 6, № 2, A 49 — A 53.
14. S n e d d o n I. N. The distribution of stress in the neighborhood of a crack in an elastic solid. Proc. Roy. Soc., 1946, A 187, 229—260.
15' S n e d d o n I. N. Fourier Transforms. McGraw Hill, N.-Y., 1951.
16' S n e d d o n I. N. and E l l i o t H. A. The opening of a Griffith crack under internal pressure. Quart. Appl. Math., 1946, 4, 262—267.
17. W i l l i a m s M. L. On the stress distribution at the base of a stationary crack J. Appl. Mech., 1957, 24, 109—114.

18. Muskhelishvili, N.I. Nekotoryye osnovnyye zadachi matematicheskoy teorii uprugosti [Certain Basic Problems of the Mathematical Theory of Elasticity], 4th Edition, Izd-vo AN SSSR [Publishing House of the Academy of Sciences USSR], Moscow-Leningrad, 1954.

19. Muskhelishvili, N.I. Singulyarnyye integral'nyye uravneniya [Singular Integral Equations], GTTI [State Press for Theoretical and Technical Literature], Moscow-Leningrad, 1946.

20. S a c k R. A. Extension of Griffith theory of rupture to three dimensions. Proc. Phys. Soc., 1946, 58, 729—736.
21. W i l l m o r e T. J. The distribution of stress in the neighborhood of a crack. Quart. Journ. Mech. Appl. Math., 1949, 2, 53—64.
22. B o w i e O. L. Analysis of an infinite plate containing radial cracks originating at the boundary of an internal circular hole. J. Math. and Phys., 1956, 25, 60—71.
23. I r w i n G. R. Fracture dynamics, in «Fracturing of Metals», ASM, Cleveland, 1948, pp. 147—166.
24. O r o w a n E. O. Fundamentals of brittle behavior of metals, in «Fatigue and Fracture of Metals» (Murray W. M., ed), Wiley, N.-Y., 1950, pp. 139—167.
25. I r w i n G. R. and K i e s J. A. Fracturing and fracture dynamics. Weld. Journ. Res. Suppl., 1952, 31, 95s—100s.
26. I r w i n G. R. and K i e s J. A. Critical energy rate analysis of fracture strength, Weld. Journ. Res. Suppl., 1954, 33, 193s—198s.
27. O r o w a n E. O. Energy criteria of fracture. Weld. Journ. Res. Suppl., 1955, 34, 157s—160s.

28. Fehlbeck D. K. and Orowan E. O. Experiments on brittle fracture of steel plates. Weld. Journ. Res. Suppl., 1955, 34, 570s—575s.
29. Wells A. A. The mechanics of notch brittle fracture. Weld. Research, 1953, 7, 34r—56r.
30. Wells A. A. The brittle fracture strength of welded steel plates. Quart. Trans. Inst. Nav. Arch., 1956, 98, 296—311.
31. Wells A. A. Strain energy release rates for fractures caused by wedge action. NRL Report № 4705, 1956.
32. Winno D. H. and Wundt B. M. Application of the Griffith — Irwin theory of crack propagation to the bursting behavior of disks including analytical and experimental studies. Trans. ASME, 1958, 80, 1643—1658.
33. Bueckner H. F. The propagation of cracks and the energy of elastic deformation.

34. Rebinder, P.A., Fiziko-khimicheskiye issledovaniya protsessov deformatsii tverdykh tel [Physicochemical Research on Deformation Processes in Solids], Yubileynyy sb. posvyashchennyy XXX-letiyu Velikoy Oktyabr'skoy revolyutsii [Anniversary Symposium Dedicated to the 30th Anniversary of the Great October Revolution], Moscow, Izd-vo AN SSSR, 1947, pp. 533-561.

35. Elliot H. A. An analysis of the conditions for rupture due to Griffith cracks. Proc. Phys. Soc., 1947, 59, 208—223.
36. Mott N. F. Fracture of metals; theoretical consideration, Engineering, 1948, 165, 16—18.

37. Rzhanitsyn, A.R., K voprosu o protsesse razrusheniya materiala pri rastyazhenii [The Process of Fracture in a Material under Tension], in the Collection Issledovaniya po voprosam stroitel'noy mekhaniki i teorii plastichnosti [Research on Problems in Structural Mechanics and the Theory of Plasticity], Moscow, Stroyizdat [State Publishing House for Literature on Construction and Architecture], 1956, pp. 66-83.

38. Zheltov, Yu.P., and Khristianovich, S.A., O mekhanizme gidravlicheskogo razriva neftenosnogo plasta [The Mechanism of Hydraulic Splitting of a Petroleum-Bearing Stratum], Izv. AN SSSR, OTN [Bulletin of the Academy of Sciences USSR, Department of Technical Sciences], 1955, No. 5, 3-41.

39. Barenblatt, G.I., and Khristianovich, S.A., Ob obrushenii krovli pri gornykh vyrabotkakh [The Caving of Cap Rock in Mining Operations], Izv. AN SSSR, OTN, 1955, No. 11, 73-86.

40. Barenblatt, G.I., O nekotorykh zadachakh teorii uprugosti, voznikayushchikh pri issledovanii mekhanizma gidravlicheskogo

razryva neftenosnogo plasta [Certain Problems of Elasticity Theory which Arise in the Study of the Mechanism of Hydraulic Fracture in a Petroleum-Bearing Stratum], PMM [Applied Mechanics and Mathematics], V. XX, 1956, 475-486.

41. Barenblatt, G.I., Ob obrazovanii gorizontal'nykh treshchin pri gidravlicheskom razryve neftenosnogo plasta [The Formation of Horizontal Cracks in the Hydraulic Fracture of a Petroleum-Bearing Stratum], Izv. AN SSSR, OTN, 1956, No. 9, 101-105.

42. Zheltov, Yu.P., Ob odnom priblizhennom metode raschetov razmerov treshchin, obrazuyushchikhsya pri gidravlicheskom pazryve plasta [One Approximate Method of Calculating the Sizes of Cracks Formed in the Hydraulic Splitting of a Stratum], Izv. AN SSSR, OTN, 1957, No. 3, 180-182.

43. Zheltov, Yu.P. Ob obrazovanii vertikal'nykh treshchin v plaste pri pomoshchi fil'truyushcheysya zhidkosti [The Formation of Vertical Cracks in a Stratum by a Liquid Filtering Through It], Izv. AN SSSR, OTN, 1957, No. 8, 56-62.

44. Westergaard H. M. Stresses at a crack, size of the crack and the bending of reinforced concrete. J. Americ. Concrete Inst., 1933, 5, № 2, 93—102.

45. Irwin G. R. Analysis of stresses and strains near the end of a crack traversing a plate. J. Appl. Mech., 1957, 24, 361—364.

46. Irwin G. R. Relation of stresses near a crack to the crack extension force. Proc. 9-th Intern. Congr. Appl. Mech. Brussels, 1957, pp. 245—251.

47. Irwin G. R. Fracture, in «Handbuch der Physik», B. VI (Flugge, S. ed.), Springer, Berlin, 1958, pp. 551—590.

48. Irwin G. R., Kies J. A. and Smith H. L. Fracture strength relative to onset and arrest of crack propagation. Proc. Americ. Soc. Test. Mater., 1958/1959, 58, 640—657.

49. Kies, J.A., Smith, H.L., Irwin, G.R., La mecanique des ruptures et son application aux travaux de l'ingenieur [The Mechanics of Fracture and its Application in the Work of the Engineer], Mem. scient. rev. metallurgie [Annals of Science, Metallurgical Review], 1960, 57, No. 2, 101-117.

50. Bueckner H. F. Some stress singularities and their computation by means of integral equations, in «Boundary Problems in Differential Equations» (Langer R.E., ed.), Univ. Wisconsin Press, 1960, pp. 215—230.

51. Irwin G. R. The crack-extension-force for a crack at a free surface boundary NRL Report, № 5120, 1958.
52. Romualdi J. P. and Sanders P. H. Fracture arrest by riveted stiffeners, Proc. 4-h Midwest. Conf. Solid Mech., Univ. Texas Press, 1959/1960, pp. 74—90.
53. Roesler F. C. Brittle fracture near equilibrium. Proc. Phys. Soc., 1956, B69, 981—992.
54. Benbow J. J. Cone cracks in fused silica. Proc. Phys. Soc., 1960, B75, 697—699.

55. Drozdovskiy, B.A., and Fridman, Ya.B., Vliyaniye treshchin na mekhanicheskiye svoystva konstruktsionnykh staley [The Influence of Cracks on the Mechanical Properties of Structural Steels], Metallurgizdat [State Scientific and Technical Publishing House for Literature on Ferrous and Nonferrous Metallurgy], Moscow, 1960.

56. Barenblat, G.I., O ravnovesnykh treshchinakh, obrazuyushchikhsya pri khrupkom razrushenii [Equilibrium Cracks Formed in Brittle Fracture], Dokl. AN SSSR [Proceedings of the Academy of Sciences USSR], 1959, 127, 47-50.

57. Barenblatt, G.I., O ravnovesnykh treshchinakh, obrazuyushchikhsya pri khrupkom razrushenii. Obshchiye predstavleniya i gipotezy. Osesimmetrichnyye treshchiny [Equilibrium Cracks Formed in Brittle Fracture. General Assumptions and Hypotheses. Axially Symmetric Cracks], PMM, Vol. XXIII, 434-444.

58. Barenblatt, G.I., O ravnovesnykh treshchinakh, obrazuyushchikhsya pri khrupkom razrushenii. Pryamolineynyye treshchiny v ploskikh plastinkakh [Equilibrium Cracks Formed in Brittle Fracture. Rectilinear Cracks in Flat Plates], PMM, 1959, Vol. XXIII, 706-721.

59. Barenblatt, G.I., O ravnovesnykh treshchinakh, obrazuyushchinkhsya pri khrupkom razrushenii. Ustoychivost' izolirovannykh treshchin [Equilibrium Cracks Formed in Brittle Fracture. Stability of Isolated Cracks], PMM, 1959, Vol. XXIII, 893-900.

60. Barenblatt, G.I., Ob usloviyakh konechnosti v mekhanike sploshnykh

sred. Staticheskiye zadachi teorii uprugosti [Conditions of Finitude in the Mechanics of Continuous Media. Static Problems of Elasticity Theory], PMM, 1960, Vol. XXIV, 316-322.

61. Barenblatt, G.I., Ob osnovnykh predstavleniyakh teorii ravnovesnykh treshchin, obrazuyushchikhsya pri khrupkom pasrushenii [Basic Assumptions of the Theory of Equilibrium Cracks formed in Brittle Fracture], Sb. rabot posvyashchennyy 70-letiyu N.I. Muskhelishvili [Collection of Works Issued in Honor of the 70th Birthday of N.I. Muskhelishvili], Izd-vo AN SSSR, Moscow-Leningrad, 1961, pp. 41-58.

62. Aleksandrov, A.P., Zhurkov, S.N., Yavleniye krupkogo razryva [The Phenomenon of Brittle Fracture], Moscow, GTTI, 1933.

63. Garber, R.I., and Gindin, I.A., Fizika prochnosti kristallicheskikh tel [Physics of the Strength of Crystalline Bodies], Usp. fizicheskikh nauk [Advances in the Physical Sciences], 1960, 70, 57-110.

64. Deryagin, B.V., Abrikosova, I.I., Lifshits, Y..M., Molekulyarnoye prityazheniye kondensirovannykh tel [Molecular Attraction in Condensed Bodies], Usp. fizicheskikh nauk, 1958, 64, 493-528.

65. Derjaguin B. V. and Abrikossova I. I. Direct measurements of molecular attraction of solids, J. Phys. Chem. Solids, 1958, 5.

66. Jongh J. G. V., de. Measurements of Retarded van der Waals-Forces Thesis, Abels, Utrecht, 1958.

67. Born M., Kun Huang. Dynamical Theory of Crystal Lattices, Clarendon Press, Oxford, 1954.

68. Leibfried, G., Gittertheorie der mechanischen und termischen Eigenschaften der Krystalle [Lattice Theory of the Mechanical and Thermal Properties of Crystals], Handbuch der Physik [Handbook of Physics], Vol. VII, I (S. Flugge, Editor), Springer, Berlin, 1955, pp. 104-324.

69. Leonov, M.Ya. and Panasyuk, V.V., Rozvitok naydribnishikh trishchin v tverdomu tili [Development of Fine Cracks in Solids],

Prikladna mekhanika [Applied Mechanics], AN USSR [Academy of Sciences, Ukrainian SSSR], 1959, 5, 391-401.

70. Panasyuk, V.V., Opredeleniye napryazheniy i deformatsiy vblizi mel'chayshey treshchiny [Determination of Stresses and Strains in the Vicinity of Minute Cracks], Nauchn. zap [Scientific Records], In-ta mashinovedeniya i avtomatiki AN USSR [Institute for Mechanical Engineering and Automatic Machinery, Academy of Sciences, Ukrainian SSR], 1960, 7, 114-127.

71. Landau, L.D., Lifshits, Ye.M., Mekhanika sploshnykh sred [Mechanics of Continuous Media], GITTL [State Publishing House for Theoretical and Technical Literature], Moscow-Leningrad, 1954.

72. Galin, L.A., Kontaktnyye zadachi teorii uprugosti [Contact Problems of Elasticity Theory], Moscow, GITTL, 1953.

73. Shtayerman, I.Ya., Kontaktnaya zadacha teorii uprugosti [The Contact Problem in Elasticity Theory], Moscow, GITTL, 1949.

74. Parker E. R. Brittle Behavior of Engineering Structures, Wiley, N. Y., 1957.

75. Davidenkov, N.N., Problema udara v metallovedenii [The Problem of Shock in Metallography], Izd-vo AN SSSR, Moscow-Leningrad, 1938.

76. Pashkov, P.O., Razryv metallov [Fracture of Metals], Sudpromgiz [State All-Union Publishing House for the Shipbuilding Industry], Leningrad, 1960.

77. Potak, Ya.M., Khrupkiye razrusheniya stali i stal'nykh detaley [Brittle Fracture of Steel and Steel Products], Oborongiz [State Scientific and Technical Publishing House for the Defense Industry], Moscow, 1955.

78. Averbach B. L. et al. (ed.) Fracture, Wiley, N. Y., 1959.

79. Sedov, L.I., Metody podobiya i razmernosti v mekhanike [Similitude and Dimensional Methods in Mechanics], 4th Edition, GITTL, Moscow, 1957.

80. Neuber, N., Kerbspannungslehre. Grudlagen fur genaue Spannungsrechnung [Notch-Stress Theory. Foundations for Exact Stress Measurements], Springer, Berlin, 1937.

81. Frankland I. M. Triaxial tension at the head of a rapidly running crack in a plate. Paper ASME, 1959, № APM-11.

82. Masubuchi K. Dislocation and strain energy release during crack propagation in residual stress field. Proc. 8-th Japan Nath. Congr. Appl. Mech., 1958/1959, pp. 147—150.

83. Stroh A. N. Dislocations and cracks in anisotropic elasticity, Phil. mag., 1958, VIII ser. 3, 625—646.

84. Barenblatt, G.I., and Cherepanov, G.P., O ravnovesii i rasprostranenii treshchin v anizotropnoy srede [Equilibrium and Propagation of Cracks in an Anisotropic Medium], PMM, 1961, Vol. XXV, 46-55.

84a. Kachanov, L.M., K kinetike rosta treshchin [Kinetics of Crack Propagation], PMM, 1961, Vol. XXV, Issue 3, p. 498.

85. Wigglesworth L. A. Stress distribution in a notched plate, Mathematika, 1957, 4, 76—96.

86. Sherman, D.I., Ob odnom metode resheniya staticheskoy ploskoy zadachi teorii uprugosti dlya mnogosvyaznykh oblastey [One Method of Solving the Static Plane Problem of Elasticity Theory for Multiply-connected Regions], Tr. Seysmologicheskogo instituta Akademii nauk SSSR [Transactions of the Seismological Institute, Academy of Sciences USSR], Izd-vo AN SSSR, 1935.

87. Mikhlin, S.G., Ploskaya zadacha teorii uprugosti [The Plane Problem of Elasticity Theory], Tr. Seysmologicheskogo instituta Akademii nauk AN SSSR, No. 65, Izd-vo AN SSSR, 1935.

88. Barenblatt, G.I., and Cherepanov, G.P., O vliyanii granits tela na razvitiye treshchin khrupkogo razrusheniya [The Influence of the Boundary of a Body on Crack Propagation in Brittle Fracture], Izv. AN SSSR, OTN, Mekhanika i mashinostroyeniye [Mechanics and Machine Building], 1960, No. 3, 79-88.

89. Koiter W. T. An infinite row of collinear cracks in an infinite elastic sheet. Ingenieur-Archiv, 1959, 28, 168—172.

90. Muskhelishvili, I.I., Osnovnyye granichnyye zadachi teorii uprugosti dlya ploskosti s pryamolineynymi pazrezami [Basic Boundary Problems of Elasticity Theory for Planes with Rectilinear Notches], Soobshch. AN GruzSSr [Communications of the Academy of Sciences Georgian SSR], 1942, 3, 103-110.

91. Sherman, D.I., Smeshannaya zadacha teorii potentsiala i teorii uprugosti dlya ploskosti s konechnym chislom pryamolineynykh pazrezov [The Mixed Problem of Potential Theory and Elasticity Theory for Planes with a Finite Number of Rectilinear Notches], Dokl. AN SSSR, 1940, 27, 330-334.

92. Wundt B. M. A unified interpretation of room-temperature strength of notched specimens as influenced by their size. Paper ASME, 1959, № 59—MET—9.

93. Yusuff S. Fracture phenomena in metal plates. Paper presented at the 10-th Intern. Congr. Appl. Mech. Stresa, 1960.

94. Ustinov, Yu.A., O vliyanii svobodnoy granitsy poluprostranstva na rasprostraneniye treshchiny [The Influence of Free Half-Space Boundaries on Crack Propagation], Izv. AN SSSR, OTN. Mekhanika i mashinostroyeniye, 1959, No. 4, 181-183.

95. Barenblatt, G.I., Cherepanov, G.P., O rasklinivanii khrupkikh tel [The Wedging of Brittle Bodies], PMM, 1960, Vol. XXIV, 667-682.

96. Muskhelishvili, N.I., Nekotoryye osnovnyye zadachi matematicheskoy teorii uprugosti [Certain Basic Problems in Mathematical Elasticity Theory], 2nd Edition, Izd-vo AN SSSR, Moscow-Leningrad, 1935.

97. Bitsadze, A.V., O mestnykh deformatsiyakh pri szhatii uprugikh tel [Local Deformation on the Compression of Elastic Bodies], Soobshch. AN GruzSSR, 1942, 3, 419-424.

98. Markuzon, I.A., O rasklinivanii khrupkogo tela klinom konechnoy dliny [Wedging a Brittle Body with a Wedge of Finite Length], PMM, 1961, Vol. XXV, 356-361.

99. Kuznetsov, I.A. Poverkhnostnaya energiya tverdykh tel [Surface energy of Solids], Moscow, GITTL, 1954.

100. Yoffe E. The moving Griffith crack, Phil. mag., 1951, VII ser., 42, 739—750.
101. Roberts D. K. and Wells A. A. The velocity of brittle fractures. Engineering, 1954, 178, 820—821.
102. Stroh A. N. A theory of the fracture of metals. Advances in Physics, 1957, 6, 418—465.
103. Craggs I. W. On the propagation of a crack in an elastic-brittle material. J. Mech. Phys. Solids, 1960, 8, 66—75.
104. Dang Dinh An. Elastic waves by a force moving along a crack. J. Math. and Phys., 1960, 38, 246—256.
105. Baker B. R. Dynamic stresses created by a moving crack. Paper presented at the 10-th Intern. Congr. Appl. Mech. Stresa, 1960.
106. Radok J. R. M. On the solutions of problems of dynamic plane elasticity. Quart. Appl. Math., 1956, 14, 289—298.
107. Broberg K. B. The propagation of a brittle crack. Paper presented at the 10-th Intern. Congr. Appl. Mech. Stresa, 1960.
108. Broberg K. B. The propagation of a brittle crack. Arkiv för Fysik, 1960, 18, 159—192.
109. Bilby B. A. and Bullough R. The formation of twins by a moving crack. Phil. mag., 1954, VII, ser. 45, 631—646.
110. Mc Clintock F. A. and Sukhatme S. P. Travelling cracks in elastic materials under longitudinal shear. J. Mech. Phys. Solids, 1960, 8, 187—193.
111. Suits J. C. Cleavage, ductility and tenacity in crystals. Discussion, in «Fracture» (B. L. Anderson et al., ed.), Wiley, N.-Y., 1959, pp. 223—224.

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